# ON SOLUTIONS IN GENERALIZED FUNCTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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## Abstract

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*MATHEMATICS*

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# ON SOLUTIONS IN GENERALIZED FUNCTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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In the paper ([1]) it was proved that an equation of the form

$$L(y) \equiv \sum_{k=0}^{m} a_k x^{k+r} y^{(k)}(x) + \sum_{q=0}^{n} b_q x^q y^{(q)}(x) = 0, \quad m > n,\ r > 0,$$

in the space of generalized functions $(S_0^\beta)'$, with some $\beta > 1$, has a fundamental system of $2m + r$ linearly independent solutions. In the present note we study the more general equation

$$L(y) \equiv \sum_{p=0}^{n} \sum_{k=0}^{m_p} a_{kp} x^{k+p} y^{(k)}(x) = 0, \tag{1}$$

$$m_n > m_p > 0, \quad p = 0, 1, \dots, n-1, \quad a_{m_n n} \neq 0, \quad a_{00} \neq 0,$$

in the same space $(S_0^\beta)'$, $\beta > 1$.

Denote by $\Omega_0$ the totality of all solutions of equation (1) in $(S_0^\beta)'$ concentrated at the point $x = 0$.

**Theorem 1.** *The dimension of the subspace $\Omega_0 \subset (S_0^\beta)'$ is equal to $n$.*

**Proof.** Let

$$y(x) = \sum_{q=0}^{\infty} c_q \delta^{(q)}(x). \tag{2}$$

Under the condition of convergence of the series (2), this expression is the general form of a functional in $(S_0^\beta)'$ concentrated at the point $x=0$ (see ([2])). To prove the theorem it is enough to determine how many free constants $c_q$ there are in the expansion (2) such that expression (2), with these free constants, is a solution of equation (1). Substituting expression (2) into equation (1) and using the formulas $x^k\delta^{(q)}(x) = (-1)^k[q]_k\delta^{(q-k)}(x)$ for $k<q$ ($=0$ for $k>q$), by virtue of the linear independence of $\delta^{(q)}(x)$ for different $q$, equating to zero the resulting coefficient of $\delta^{(q)}(x)$, we obtain

$$\sum_{p=0}^{n} D_p(q)c_{q+p} = 0, \quad q=0,1,\dots, \tag{3}$$

where

$$D_p(q) = \sum_{k=0}^{m_p} a_{kp}(-1)^{k+p}[q+k+p]_{k+p}, \quad [q]_\rho = q(q-1)\cdots(q-\rho+1).$$

We denote the degree of the polynomial $D_p(q)$ by $d_p$. Obviously, $d_p = m_p + p$, and, by virtue of our assumptions, $d_n > d_p,\ p=0,1,\dots,n-1$. It is clear that in equation (3), assigning $c_0, c_1, \dots, c_{n-1}$ arbitrarily, it is easy to determine subsequent $c_q$ such that

$$c_{q+n} = -\sum_{p=0}^{n-1}\frac{D_p(q)}{D_n(q)}\,c_{q+p}, \qquad q=0,1,2,\dots \tag{4}$$

It remains for us to verify that the series (2) constructed from the coefficients (4) thus obtained converges in $(S_0^\beta)'$ with some $\beta>1$. For this it is sufficient that, as $q\to\infty$, the coefficients $c_q$ satisfy the estimates

$$|c_q| \le \frac{B^q}{q^{qr_0}}\,M_q, \qquad r_0>1, \quad B>1, \quad M_q<c. \tag{5}$$

Indeed, under such an estimate for $c_q$ we have

$$\left|\sum_{q=0}^{\infty}(c_q\delta^{(q)}(x),\varphi(x))\right| = \left|\sum_{q=0}^{\infty}c_q(-1)^q\varphi^{(q)}(0)\right| \le \sum_{q=0}^{\infty}|c_q|\,|\varphi^{(q)}(0)| \le$$

$$\le \sum_{q=0}^{\infty}\frac{B^qB_1^q q^{q\beta}}{q^{qr_0}}\,M_q \le \sum_{q=0}^{\infty}\left(\frac{\bar B}{q^{r_0-\beta}}\right)^q < \infty \qquad \text{for } \beta<r_0.$$

Let us prove the estimates (5). Denoting $M_q = c_q q^{qr_0}/B^q$, from (4) we obtain

$$M_{q+n} = -\frac{1}{B^{q+n}} \sum_{p=0}^{n-1} \frac{D_p(q)}{D_n(q)} \frac{B^q (q+n)^{(q+n)r_0}}{(q+p)^{(q+p)r_0}} M_{q+p}.$$

For

$$r_0 = \min_{0<p<n-1} \left\{ 1 + \frac{m_n - m_p}{n-p} \right\}$$

there exists such a $q_1$ that for $q > q_1$

$$M_{q+n} \leq \frac{c}{(1+q)^{\varepsilon+1} B} \sum_{p=0}^{n-1} M_{q+p}.$$

Let $\bar{M}_{n-1} = \max(M_0, M_1, \dots, M_{n-1})$; then

$$M_{q+n} \leq \frac{c_n}{(1+q)^{\varepsilon}} \frac{1}{B} \bar{M}_{q+n-1};$$

choosing $B > cn$, we obtain $M_{q+n} \leq \bar{M}_{q+n} \leq \bar{M}_{q+n-1}$, which means boundedness of $M_q$ as $q \to \infty$. This proves Theorem 1 completely.

Put

$$f^{\lambda}(x) = |x|^{\lambda} \Big/ \Gamma\left(\frac{\lambda+1}{2}\right), \qquad g^{\lambda}(x) = |x|^{\lambda} \operatorname{sign} x \Big/ \Gamma\left(\frac{\lambda+2}{2}\right).$$

As is known ([3], p. 81 and following), by analytic continuation these functions define a functional on $S' \supset (S_0^{\beta})'$ for all values of $\lambda$; in particular, for $\lambda = -(2n+1)$ and $\lambda = -2n$ we obtain

$$f^{-(2n+1)}(x) = (-1)^n n!\, \delta^{(2n)}(x)/(2n)!, \qquad g^{-2n}(x) = \frac{(-1)^n (n-1)!\, \delta^{(2n-1)}(x)}{(2n-1)!}.$$

The following formulas for differentiating and multiplying by powers of $x$ the functionals $f^{\lambda}(x)$ and $g^{\lambda}(x)$ hold:

$$x^q [f^{\lambda}(x)]^{(r)} = \begin{cases} 2^r \left[\frac{\lambda}{2}\right]_{E(r/2+1)} \left[\frac{\lambda+q-r-1}{2}\right]_{E(q/2)} f^{\lambda+q-r}(x), & \text{if } q+r \text{ is even,} \\ 2^r \left[\frac{\lambda}{2}\right]_{E(r/2+1)} \left[\frac{\lambda+q-r}{2}\right]_{E(q/2+1)} g^{\lambda+q-r}(x), & \text{if } q+r \text{ is odd;} \end{cases} \tag{6}$$

$$x^q[g^\lambda(x)]^{(r)} = \begin{cases} 2^r \left[\dfrac{\lambda-1}{2}\right]_{E(r/2)} \left[\dfrac{\lambda+q-r}{2}\right]_{E(q/2+1)} g^{\lambda+q-r}(x), & \text{if } q+r \text{ is even,} \\ 2^r \left[\dfrac{\lambda-1}{2}\right]_{E(r/2)} \left[\dfrac{\lambda+q-r-1}{2}\right]_{E(q/2)} f^{\lambda+q-r}(x), & \text{if } q+r \text{ is odd.} \end{cases} \tag{7}$$

Here $E(\lambda)$ is the integer part of $\lambda$.

Put

$$R_p(\lambda) = \sum_{k=0}^{m_p} a_{kp}[\lambda-p]_k, \qquad A_p(\lambda) = \left[\frac{\lambda+1}{2}\right]_{E(p/2)} R_p(\lambda),$$

$$B_p(\lambda) = \left[\frac{\lambda}{2}\right]_{E(p/2+1)} R_p(\lambda).$$

We shall call the polynomial $A_n(\lambda)$ the characteristic polynomial of equation (1).

**Definition.** We shall say that the polynomial $A(\lambda)$ has no generalized multiple roots modulo $r$, if the arithmetic progressions with difference $r$ constructed on the roots of the polynomial $A(\lambda)$ do not intersect. If, however, the arithmetic progression with difference $r$ constructed over the root $\lambda_1$ contains $k$ roots of the polynomial $A(\lambda)$, then we shall say that $\lambda_1$ is a generalized multiple root modulo $r$ of the polynomial $A(\lambda)$ of order $k$.

**Theorem 2.** If the polynomial $A_n(\lambda)$ has no generalized multiple roots modulo $n$ for even $n$ and modulo $n-1$ for odd $n$, then equation (1) in the space $(S_0^\beta)'$ with some $\beta > 1$ has $2m_n + n$ linearly independent solutions of the form

$$y(x) = \sum_\lambda \xi_\lambda f^\lambda(x) + \sum_\lambda \eta_\lambda g^\lambda(x), \tag{8}$$

where the summation index $\lambda$ runs through the values from a certain arithmetic progression with difference $n$ for even $n$ and with difference $n-1$ for odd $n$.

**Proof.** Substituting the expressions (8) into equation (1) and using formulas (6), (7), in view of the linear independence of $f^\lambda(x)$ and $g^\lambda(x)$ for different $\lambda$, we obtain

$$\sum_{p=0}^{n}{}'' A_p(\lambda)\xi_{\lambda-p} + \sum_{p=0}^{n}{}' A_p(\lambda)\eta_{\lambda-p} = 0, \tag{9}$$

$$\sum_{p=0}^{n}{}'' B_p(\lambda)\eta_{\lambda-p} + \sum_{p=0}^{n}{}' B_p(\lambda)\xi_{\lambda-p} = 0. \tag{10}$$

Here two primes and one prime over $\sum$ mean that the summation extends over even and odd $p$, respectively. The pair of equations (3), (9) is equivalent to equation (1). Suppose that $n$ is even. In this case the degrees $a_n$ and $b_n$ of the polynomials $A_n(\lambda)$ and $B_n(\lambda)$ are equal to each other, namely $a_n = b_n = (n/2 + m_n)$.

Using the structure of the polynomials $A_p(\lambda)$ and $B_p(\lambda)$ and substituting in (9), (10) the values $\lambda = 1, 3, 5, \dots, 0, 2, 4, \dots$, it is not difficult to see that $\xi_{2k+1} = \eta_{2k} = 0$ for the values $k = 0, 1, 2, \dots$. Further, assigning $\xi_{-1}, \xi_{-3}, \dots, \xi_{-(n-1)}$ and $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$ arbitrarily, it is easy to determine all values $\xi_k$ lying on arithmetic progressions with difference $n$ constructed over the numbers $\xi_{-1}, \xi_{-3}, \dots, \xi_{-(n-1)}$, and all values $\eta_k$ lying on arithmetic progressions with difference $n$ constructed over the numbers $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$. The $\xi_k$ and $\eta_k$ found in this way depend only on the values $\xi_{-1}, \xi_{-3}, \xi_{-5}, \dots, \xi_{-(n-1)}$; $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$ and determine $n$ linearly independent solutions of equation (1). These solutions are concentrated at the point $x = 0$.

Let now $\lambda$ be a root of the polynomial $R_n(\lambda)$. Assigning $\xi_{\lambda_1-n}, \eta_{\lambda_1-n}$ arbitrarily and $\xi_{\lambda_1-p-kn} = \eta_{\lambda_1-p-kn} = 0$ for the values $p = 0, 1, \dots, n-1$, $k = 0, 1, 2, \dots$, we find all values $\xi_\lambda$ and $\eta_\lambda$ belonging to arithme-

tic progression constructed above $\lambda_1$ with difference $n$. In this case we obtain

$$\xi_{\lambda_1-kn} \le \frac{A_0(\lambda_1-(k-1)n)}{A_n(\lambda_1-(k-1)n)}\,\xi_{\lambda_1-(k-1)n},$$

whence

$$\left|\xi_{\lambda_1-kn}\right| \le \frac{c}{k^{\,k(m_n+n+m_0)}}\,\xi_{\lambda_1-n}. \tag{11}$$

Similarly,

$$\left|\eta_{\lambda_1-kn}\right| \le \frac{c}{k^{\,k(m_n+n-m_0)}}\,\eta_{\lambda_1-n}. \tag{12}$$

The values $\xi_\lambda$ and $\eta_\lambda$ thus found determine, by formula (8), two solutions of equation (1). Since $A_n(\lambda)$ has no multiple generalized roots, these solutions are linearly independent. The convergence of the series (8) with these $\xi_\lambda$ and $\eta_\lambda$ follows from the estimates (11), (12). Carrying out the indicated procedure for all roots of the polynomial $R_n(\lambda)$, we obtain another $2m_n$ linearly independent solutions of equation (1).

Thus, for even $n$, equation (1) has $2m_n + n$ linearly independent solutions of the form (8), of which $n$ are concentrated at the point $x = 0$.

For odd $n$ the proof is completely analogous, except that in this case $a_n = (n-1)/2 + m_n$, $b_n = (n+1)/2 + m_n$, and the arithmetic progressions will be constructed not with difference $n$, but with difference $n-1$. This completes the proof of Theorem 2.

**Theorem 3.** *The linearly independent solutions found in Theorem 2 form a fundamental system of solutions of equation (1) in* $(S_0^\beta)'$, $\beta > 1$.

The proof of Theorem 3 is carried out in the same way as the proof of the analogous assertion (Theorems 2 and 3) in paper ([1]).

**Example.** The equation

$$x^3 y' + \left({}^3\!/_2 x^2 + x + 1\right) y = 0$$

has in $(S_0^{3/2})'$ the fundamental system of solutions

$$y_1(x) = \delta(x) + \frac{1}{5}\delta''(x) - \frac{4}{7 \cdot 5 \cdot 3!}\delta^{\mathrm{IV}}(x) + \cdots$$

$$y_2(x) = -\delta' + \frac{1}{5}\delta'''(x) - \frac{2}{5 \cdot 3!}\delta^{\mathrm{V}}(x) + \cdots$$

$$y_3(x) = \sum_{k=0}^{\infty} \frac{2^k (-1)^k}{k! \prod_{\mu=1}^{k} (4\mu - 1)} f^{-(1+4k)/2}(x),$$

$$y_4(x) = \sum_{n=0}^{\infty} \frac{(-1)^k \, 2k}{k! \prod_{\mu=0}^{n} (4\mu + 1)} g^{-(1+4k)/2}(x).$$

The solutions $y_1$ and $y_2$ are concentrated at the point $x = 0$.

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## References

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