ON THE APPROXIMATION OF A PROBLEM OF OPTIMAL CONTROL IN A SYSTEM WITH AFTEREFFECT

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Abstract

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ON THE APPROXIMATION OF A PROBLEM OF OPTIMAL CONTROL IN A SYSTEM WITH AFTEREFFECT

Consider the problem ([1,2]) of optimal control $u^0[t, x(t+\vartheta)]$, which stabilizes, up to asymptotic stability, the system with aftereffect

$$\dot{x}(t) = \int_{-\sigma}^{0} d_\vartheta H(t,\vartheta)x(t+\vartheta) + B(t)u \tag{1}$$

and at the same time minimizes the integral

$$I_\infty[t_0, x_0(\vartheta), u] = \int_{t_0}^{\infty} [\|x(t)\|^2 + \|u(t)\|^2]\, dt. \tag{2}$$

Here $x = \{x_i\}$ is an n-dimensional vector of phase coordinates of the controlled object; $x(t+\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$ is an element of the trajectory of system (1) of temporal length $\sigma > 0$—const; $H(t,\vartheta)$ is a matrix with bounded variation in ϑ for $-\sigma \leq \vartheta \leq 0$; $B(t)$ is a matrix bounded for $t \geq 0$; u is an r-dimensional vector of the control force; $x_0(\vartheta)$ is the initial perturbation $x(t_0+\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$ of the object; the symbols $\|x\|$ and $\|u\|$ denote the Euclidean norms of the vectors x and u.

The optimal control $u^0[t, x(t+\vartheta)]$ for problem (1), (2) has the form of a linear functional ([2,3])

$$u^0[t, x(t+\vartheta)] = P(t)x(t) + \int_{-\sigma}^{0} Q(t,\vartheta)x(t+\vartheta)d\vartheta,$$

the concrete computation of which, however, is associated with difficulties. Therefore there arises the problem of approximating problem (1), (2) by a suitable finite-dimensional problem. One such approximation was proposed in ([4]) and investigated in the case of the proper integral (2) and the stationary system (1) in ([5]). The results given below generalize the results from article ([5]).

Let certain finite-dimensional systems be associated with system (1)

$$\dot{y}^{(m)}(t) = A^{(m)}(t)y^{m}(t) + B^{(m)}(t)v, \tag{3}$$

and let integral (2) be approximated by the integrals

$$I_{\infty}^{(m)}[t_0, y_0^{(m)}, v] = \int_{t_0}^{\infty} \left[\|\xi^{(m)}(t)\|^2 + \|v(t)\|^2\right] dt. \tag{4}$$

Here $y^{(m)}$ are k_m-dimensional vectors; $A^{(m)}(t)$ and $B^{(m)}(t)$ are, respectively, $(k_m \times k_m)$- and $(k_m \times r)$-matrices; v is an r-vector; $\xi^{(m)}$ is an n-dimensional vector that is a linear function of $y^{(m)}$.

We shall consider the motion of system (1) in the space of functions $x(\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$, taking as an element of motion the function $x(t+\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$. The right-hand side of (1) is defined for functions continuous in ϑ

functions $x(t+\vartheta)$. However, many equations (1) retain meaning also in spaces broader than the space $C[-\sigma, 0]$ of continuous functions $x(\vartheta)$. We shall therefore assume that some space X of functions $x(\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$ has been chosen (possibly broader than $C(-\sigma, 0]$), with norm $\rho[x(\vartheta)]$, where the solutions of equation (1) considered below exist. We shall also assume that in the spaces $Y^{(m)}$ of vectors $y^{(m)}$ certain norms $\rho^{(m)}[y^{(m)}]$ have been chosen. Systems (1) and (3) are connected by a linear function $y^{(m)} = f^{(m)}[x(\vartheta)]$, defined on X. The problem consists in estimating the approximation of problem (1), (2) by problem (3), (4). The central question is the following: for large m, does the control of system (1) according to the law $u = v^{(m)0}[t, f^{(m)}[x(t+\vartheta)]]$ give an effect close to the optimal one given by the control $u = u^0[t, x(t+\vartheta)]$.

We introduce the following notation. Let $x(t, t_0, x_0(\vartheta), u)$ and $y^{(m)}(t, t_0, y_0^{(m)}, v)$ be, respectively, the motions of systems (1) and (3) generated by the initial conditions $x(t_0+\vartheta) = x_0(\vartheta)$ $(-\sigma \leq \vartheta \leq 0)$, $y^{(m)}(t_0) = y_0^{(m)}$, under certain chosen controls $u(t)$ and $v(t)$, or $u[t, x(t+\vartheta)]$, $v[t, y^{(m)}(t)]$. By the symbol $\xi^{(m)}(t, t_0, y_0^{(m)}, v)$ we denote the functions $\xi^{(m)}[y^{(m)}(t, t_0, y_0^{(m)}, v)]$. We denote optimal controls for problems (1), (2) and (3), (4), respectively, by the symbols $u^0[t, x(t+\vartheta)]$ and $v^{(m)0}[t, y^{(m)}(t)]$. The same controls, but considered as functions of time $t \geq t_0$ on the motions $x(t, t_0, x_0(\vartheta), u^0)$ and $y^{(m)}(t, t_0, y_0^{(m)}, v^{(m)0})$, will be denoted by $w^0(t, t_0, x_0(\vartheta))$ and $w^{(m)0}(t, t_0, y_0^{(m)})$. We shall denote the response of system (1) to the impulse control $u_k(t) = \delta(t-t_0)$, $u_j(t) \equiv 0$ for $j \neq k$ ($\delta(t)$ is the delta function), by the symbol $h[t, t_0]_{(k)}$; the response of the motion $\xi^{(m)}(t)$ of system (3) to the action $v_k(t) = \delta(t-t_0)$, $v_j(t) \equiv 0$ for $j \neq k$, will be denoted by $h^{(m)}[t, t_0]_{(k)}$. The vectors under consideration are interpreted as column vectors. The upper index asterisk denotes transposition. Suppose that s-vector functions $w(t)$ are considered on the interval $[\alpha, \beta]$. We shall denote by the symbols $\mathcal{L}_s^2[\alpha, \beta]$, $\mathcal{L}_s^2[\alpha, \beta]$, and $C_s[\alpha, \beta]$, respectively, the functional spaces with norms

$$\|w(t)\|_{[\alpha,\beta]} = \left[\int_\alpha^\beta \|w(t)\|^2\, dt\right]^{1/2},$$

$$\|w(t)\|_{[\alpha,\beta]} = \left[\|w(\beta)\|^2 + \int_\alpha^\beta \|w(t)\|^2\, dt\right]^{1/2},$$

$$\|w(t)\|^C_{[\alpha,\beta]} = \sup[\|w(t)\| \text{ for } \alpha \le t \le \beta].$$

We assume that X coincides with $\mathcal{L}_n^2[-\sigma, 0]$. This choice is convenient for many problems. In the case of choosing another space X, corresponding changes are introduced into the conditions below. Let us formulate the conditions imposed on systems (1) and (3).

Condition 1. For every $\tau > 0$, the functions $h^{(m)}[t, t_0]_{(k)}$ and $\xi^{(m)}(t, t_0, f^{(m)}[x_0(\vartheta)], v \equiv 0)$ converge to the functions $h[t, t_0]_{(k)}$ and $x(t, t_0, x_0(\vartheta), u \equiv 0)$, respectively, in the metric of $\mathcal{L}_n^2[t_0, T)$ for all $t_0 \ge 0$ and $x_0(\vartheta) \in X_0$ (X_0 is a subspace in X with norm $\rho_0[x_0(\vartheta)]$). Moreover, for $\tau \ge \tau_0 = \text{const}$, the convergence of $h^m(t)$ and $\xi^{(m)}(t)$ to $h(t)$ and $x(t)$ also takes place in $C[t_0 + \tau_0, t_0 + \tau]$. Both convergences are uniform with respect to $t_0 \ge 0$ and $\rho_0[x_0(\vartheta)] \le 1$.

Condition 2. Let γ be any fixed positive number. If $\rho^{(m)}[y_0^{(m)}] \le \gamma$ and $\|\xi^{(m)}(t)\|_{[t_0,T)} \le \gamma$, then for any $\varepsilon > 0$ one can indicate $\delta > 0$ and N such that

$$\rho^{(m)}[y^{(m)}(T, t_0, y_0^{(m)})] < \varepsilon,$$

provided only that

$$\|\xi^{(m)}(t, t_0, y_0^{(m)}, v)\|^C_{[T-\beta,T]} \le \delta$$

and $m \ge N$. Here $\beta > 0$ is a constant independent of γ, $T \ge t_0 + \beta$.

Condition 3. Systems (1) and (3) are uniformly stabilizable for all sufficiently large values of m, i.e., there exist numbers μ and N satisfying the condition: whatever the initial data $t_0 \geqslant 0$, $x_0(\vartheta) \in X$ with $\rho[x_0(\vartheta)] \leqslant 1$, and $y_0^{(m)} \in Y^{(m)}$ with $\rho^{(m)}[y_0^{(m)}] \leqslant 1$, and $m \geqslant N$, there exist controls $u_*(t, t_0, x_0)$ and $v_*^{(m)}(t, t_0, y_0^{(m)})$, for which

$$I[t_0, x_0(\vartheta), u_*] \leqslant \mu \quad \text{and} \quad I^{(m)}[t_0, y_0^{(m)}, v_*^{(m)}] \leqslant \mu.$$

Suppose conditions 1–3 are fulfilled. Then the following assertions are valid.

Lemma 1. Uniformly with respect to $t_0 \geqslant 0$ and $\rho_0[x_0(\vartheta)] \leqslant 1$,

$$\lim_{m\to\infty} I_\infty^{(m)}[t_0, f^{(m)}[x_0(\vartheta)], v^{(m)0}] = I_\infty[t_0, x_0(\vartheta), u^0]. \tag{5}$$

A limiting relation of the form (5) is first proved for proper integrals, $\lim I_T^{(m)} = I_T$, on the basis of condition 1. Then, on the basis of conditions 2 and 3, this relation is extended to the improper integrals I_∞^m.

Lemma 2. Uniformly with respect to $t_0 \geqslant 0$ and $\rho_0[x_0(\vartheta)] \leqslant 1$,

$$\lim_{m\to\infty} \left\| w^{(m)0}(t, t_0, f^{(m)}[x_0(\vartheta)]) - w^0(t, t_0, x_0(\vartheta)) \right\|_{[t_0,\infty)} = 0. \tag{6}$$

The limiting relation (6) is proved from Lemma 1 and from an estimate of the second variation of the functional $I_\infty[t_0, x_0(\vartheta), u]$, generated by the variation $\delta u = w^{(m)0}(t) - w^0(t)$.

Theorem 1. Uniformly with respect to $t \geqslant 0$ and $\rho_0[x(\vartheta)] \leqslant 1$,

$$\lim_{m\to\infty} \left\| v^{(m)0}[t, f^{(m)}[x(\vartheta)]] - u^0[t, x(\vartheta)] \right\| = 0. \tag{7}$$

The theorem is proved from Lemma 2 and from the integral equations (5), which are satisfied by the optimal controls $w_j^0(t)$ and $w_j^{(m)0}(t)$ $(j = 1, \dots, r)$:

$$w_j^0(t) = -\int_t^\infty h^*[\tau, t]_{(j)} \left(\int_{t_0}^\tau \left[\sum_{k=1}^r h[\tau, \xi]_{(k)} w_k^0(\xi) \right] d\xi + x(\tau, t_0, x_0(\vartheta), u \equiv 0) \right) d\tau, \tag{8}$$

$$w_j^{(m)0}(t) = -\int_t^\infty h^{(m)*}[\tau, t]_{(j)} \left(\int_{t_0}^\tau \left[\sum_{k=1}^r h^{(m)}[\tau, \xi]_{[k]} w_k^{(m)0}(\xi) \right] d\xi + \xi^{(m)}(\tau, t_0, f^{(m)}[x_0(\vartheta)], v \equiv 0) \right) d\tau. \tag{9}$$

Here it is taken into account that, under conditions 1–3, the convergence of the integrals on the right-hand sides of (8) and (9) is sufficiently uniform for it to be possible, for large m, to approximate them by proper integrals.

If the control $u = u^0[t, x(t+\vartheta)]$ uniformly stabilizes system (1) in X_0, and the control $u = v^{(m)0}[t, f^{(m)}[x(t+\vartheta)]]$, for $x_0(\vartheta) \in X_0$, does not take the motion $x(t+\vartheta)$ out of X_0 for $t \geqslant t_0$, and moreover

$$\|\rho_0[x(t, t_0, x_0(\vartheta), u) - x(t, t_0, x_0(\vartheta), v)]\| \leqslant \|u - v\|\gamma$$

for $t = t_0$, then from Theorem 1 there follow the limiting relations

$$\lim_{m\to\infty} I[t_0, x_0(\vartheta), v^{(m)0}[t, f^{(m)}[x(t+\vartheta)]]] = I[t_0, x_0(\vartheta), u^0],$$

$$\lim_{m\to\infty} \left\| x(t, t_0, x_0(\vartheta), v^{(m)0}) - x(t, t_0, x_0(\vartheta), u^0) \right\|^c_{[t_0 \infty]} = 0,$$

which hold uniformly with respect to $t_0 \geqslant 0$ and $\rho_0[x_0(\vartheta)] \leqslant 1$.

For the approximation considered in article (5), conditions 1–3 are fulfilled, for example, for $X_0 = C[-\sigma, 0]$.

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Note: Figure translations are in progress. See original paper for figures.