ON A PROBLEM OF BIOPHYSICS

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Abstract

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MATHEMATICS

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ON A PROBLEM OF BIOPHYSICS

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The investigation of the question of the distribution of concentrations of substances participating in the vital processes of a living cell leads (see ([1])) to the problem for a system of diffusion equations

$$a_{11}\Delta u_{11}-(c_2+c_3)u_{11}-\partial u_{11}/\partial t=0,\qquad a_{21}\Delta u_{21}+c_2u_{11}-\partial u_{21}/\partial t=0,$$
$$(x,t)\in D_T^{(1)},$$
$$a_{k2}\Delta u_{k2}-\partial u_{k2}/\partial t=0\quad(k=1,2),\qquad (x,t)\in D_T^{(2)} \tag{1}$$

with initial conditions for $t=0$ for $u_{kl}(x,t)$, linear boundary conditions for u_{k2} on $\Gamma^{(2)}$—the outer boundary of the external medium $D_T^{(2)}$—and conjugation conditions on the surface $\Gamma^{(1)}$ of the cell membrane $D_T^{(1)}$, immersed in the medium $D_T^{(2)}$:

$$(-1)^l a_{kl}\partial u_{kl}(x,t)/\partial N_l(x,t)-h_k(u_{k2}(x,t)-u_{k1}(x,t))=0,\qquad k,l=1,2,\quad (x,t)\in\Gamma^{(1)}. \tag{2}$$

Here $u_{kl}(x,t)$ in (1) is the concentration of the substance S_k inside ($l=1$) and outside ($l=2$) the cell. S_1 is the substance entering the cell $D_T^{(1)}$ from the external medium $D_T^{(2)}$ and supplying the organism with energy. Under enzymatic transformations, part of S_1 passes into low-molecular substances S_2, which are removed from the cell, while from another part of S_1 the cell synthesizes its specific substances S_3. The constants $c_i>0$ characterize the ability of the enzymes of the cell biomass to transform S_1 into S_i for $i=2,3$. $a_{kl}>0$ ($k,l=1,2$) is the diffusion coefficient of the substance S_k inside ($l=1$) and outside ($l=2$) the cell. The constant $h_k>0$ in (2) characterizes the permeability of the cell membrane $\Gamma^{(1)}$ for the substance S_k ($k=1,2$). $N_l(x,t)$ is the normal, internal with respect to $D_T^{(l)}$, to the section $\Gamma^{(1)}\cap\{t=t>0\}$.

We shall consider a more general problem for a system of parabolic equations with discontinuous coefficients of a special kind, preserving the specificity of the system (1)–(2):

$$\sum_{i,j=1}^{n} a_{ij}^{(kl)}(x,t)\frac{\partial^2 u_{kl}}{\partial x_i \partial x_j} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}^{(kl)}(x,t)\frac{\partial u_{il}}{\partial x_j} + \sum_{i=1}^{m} c_i^{(kl)}(x,t)u_{il} - \partial u_{kl}/\partial t = f_{kl}(x,t), \quad k=1,2,\dots,m;\ l=1,2, \tag{3}$$

with supplementary conditions

$$u_{kl}(x,0) = f_{kl}^{(1)}(x), \qquad x \in \Omega^{(l)} = \overline{D_T^{(l)}} \cap \{t=0\}, \qquad k=1,2,\dots,m;\ l=1,2; \tag{4}$$

$$a_k^{(k)}(x,t)\frac{\partial u_{k2}(x,t)}{\partial \nu_k(x,t)} - b_k^{(k)}(x,t)u_{k2}(x,t) = \sum_{i\neq k,\, i=1}^{m}\left(-a_i^{(k)}(x,t)\frac{\partial u_{i2}(x,t)}{\partial \nu_i(x,t)} + b_i^{(k)}(x,t)u_{i2}(x,t)\right) + f_k^{(2)}(x,t), \quad (x,t) \in \tag{5}$$

$$(-1)^l d_{kl}^{(kl)}(x,t)\frac{\partial u_{kl}(x,t)}{\partial \nu_{kl}(x,t)} - h_{kl}^{(kl)}(x,t)(u_{k2}(x,t) - u_{k1}(x,t)) =$$

$$= \sum_{i\neq k,\, i=1}^{m}\left[(-1)^{l+1}d_{il}^{(k)}(x,t)\frac{\partial u_{il}(x,t)}{\partial \nu_{il}(x,t)} + \sum_{j=1}^{2} h_{ij}^{(kl)}(x,t)u_{ij}(x,t)\right] + f_{kl}^{(3)}(x,t), \tag{6}$$

$$(x,t)\in\Gamma^{(1)}, \qquad k=1,2,\dots,m;\ l=1,2.$$

Throughout what follows we shall use the notation and definitions of works ([2–4]). $D_T^{(1)} \cup \Gamma^{(1)} \cup D_T^{(2)} = D_T$ is a bounded domain of $(n+1)$-dimensional Euclidean space $(x,t) = (x_1, x_2, \dots, x_n, t)$, lying between the two hyperplanes $t=0$ and $t=T>0$, having as its lower base the domain $\Omega^{(1)} \cup \Omega^{(2)}$ and as its lateral boundary the closed surface $\Gamma^{(2)}$; moreover $D_T^{(1)}$ is an interior subdomain of D_T with lateral boundary surface $\Gamma^{(1)}$, separating $D_T^{(1)}$ and $D_T^{(2)}$. The closed surfaces $\Gamma^{(l)}$ $(l=1,2)$, of Lyapunov type, the tangent planes to which are nowhere orthogonal to the axis Ot, are situated between the hyperplanes $t=0$ and $t=T$ and do not intersect one another, being separated from each other by a distance not less than $d>0$. On $\Gamma^{(1)}$ there are prescribed fields of directions with unit vectors $\nu_{il}(x,t)$ $(i=1,2,\dots,m;\ l=1,2)$, lying in the section $\Omega_t^{(l)} = D_T^{(l)} \cap \{\tau = t\}$ and forming acute angles not exceeding $\pi/2 - d_0$ $(d_0>0)$ with the normal $N_{1l}(x,t)$, interior with respect to $\Omega_t^{(l)}$, at the point

$(x, t) \in \Gamma^{(1)}$. On $\Gamma^{(2)}$ there are prescribed fields of directions with unit vectors $\nu_i(x, t)$ $(i = 1, 2, \dots, m)$, lying in the section $\Omega_t^{(2)}$ and forming acute angles not exceeding $\pi/2 - d_0$ with the interior normal $N_2(x, t)$ at the point $(x, t) \in \Gamma^{(2)}$.

In conditions (5), (6) the inequalities are always assumed to hold

$$a_k^{(k)}(x, t) \geq a_0 > 0, \quad b_k^{(k)}(\bar{x}, t) \geq 0, \quad d_{kl}^{(k)}(\bar{x}, t) \geq \delta > 0, \quad h_{kl}^{(kl)}(\bar{x}, t) \geq 0,$$

$$(x, t) \in \Gamma^{(2)}, \quad (\bar{x}, t) \in \Gamma^{(1)}, \quad k = 1, 2, \dots, m; \quad l = 1, 2.$$

§ 1. Suppose that for (3)–(6) the additional conditions are satisfied

$$b_{ij}^{(kl)}(x, t) \equiv c_i^{(kl)}(x, t) \equiv 0 \text{ in } (3), \quad a_i^{(k)}(x, t) \equiv b_i^{(k)}(x, t) \equiv 0 \text{ in } (5),$$

$$d_{il}^{(k)}(x, t) \equiv h_{ij}^{(kl)}(x, t) \equiv 0 \text{ in } (6) \quad \text{for } i = k + 1, \dots, m. \tag{7}$$

With the aid of the maximum principle and Vyborny' s theorem ([5]) one proves the uniqueness of the solution of the system (3)–(7).

Theorem 1. *Suppose that for problem (3)–(7) the following conditions are satisfied: 1) all coefficients and functions entering (3)–(6) are continuous in their domains of definition, and*

$$c_k^{(kl)}(x, t) \leq c < +\infty, \quad (x, t) \in \overline{D}_T^{(l)}, \quad k = 1, 2, \dots, m; \quad l = 1, 2;$$

2) equations (3) are uniformly parabolic in $\overline{D}_T^{(l)}$ with parabolicity constant $M_0 >$ 0.

Then the solution of problem (3)–(7) is unique in the class of functions $u_{kl}(x, t)$, $k = 1, 2, \dots, m$; $l = 1, 2$, satisfying (3)–(7) and continuous on $\overline{D}_T^{(l)}$.

If one somewhat narrows the class of systems (3)–(7) by the conditions

$$b_{ij}^{(kl)}(x, t) \equiv 0 \text{ in } (3); \quad a_i^{(k)}(x, t) \equiv 0 \text{ in } (5); \quad d_{il}^{(k)}(x, t) \equiv 0 \text{ in } (6) \tag{8}$$

$$\text{for } i = 1, 2, \dots, k - 1,$$

then we obtain a class of systems possessing a number of interesting properties, following from the maximum principle, and allowing one, by means of the methods of the author and V. N. Maslennikova ([6]), to give, in particular, an a priori estimate for the maximum modulus of the solution of problem (3)–(8).

Theorem 2. *Suppose that for the solution of problem (3)–(8) conditions 1), 2) of Theorem 1 are satisfied and, in addition: 3) the coefficients of equations (3) belong to the class* $H^{0,\alpha,\alpha/2}(\overline{D}_T^{(l)})$ *(see* [3]*), their maxima of moduli being bounded by a constant* A*, and, moreover,*

$$c_k^{(kl)}(x,t) \le c < 0, \quad (x,t) \in \overline{D}_T^{(l)};$$

4) the functions entering the right-hand sides of conditions (5), (6) are bounded in modulus by a constant B*; 5) the functions entering the left-hand sides of (5), (6) belong to the class* $H_{1,\alpha,\alpha/2}^{0,1,(1+\alpha)/2}$*; 6) the surfaces* $\Gamma^{(l)}$ $(l = 1,2)$ *are of class* $\Pi_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}$, $0 < \alpha \le 1$.

Then for the solution $u_{kl}(x,t)$ *of problem (3)–(8), continuous on* $\overline{D}_T^{(l)}$*, the estimate*

$$|u_{kl}|_0^{D_T^{(l)}} \le C(d, d_0, \delta, a_0, A, B, c),$$

holds, where $C(...) > 0$ *is a constant.*

§ 2. Existence of a solution of (3)–(7).

Theorem 3. Suppose that for problem (3)–(7) conditions 1), 2) of Theorem 1 are satisfied. Suppose, in addition: 7) the surfaces $\Gamma^{(l)}$ $(l = 1,2)$ are of class $L_{1,1,(1+\beta)/2}^{1,\beta,\beta/2}$ $(0 < \alpha < \beta \leqslant 1)$; 8) the coefficients in (3) have an (α)-norm (see (2^4)) bounded by the constant M_1; 9) the initial functions from (4) have bounded $(2+\alpha)$-norms, and the coefficients entering into (5), (6) have $(1+\alpha)$-norms bounded by the constant M_2; 10) the initial conditions (4) and boundary conditions (5), (6) are compatible by virtue of (3) on the edges $\Gamma^{(s)} \cap \Omega^{(l)}$ $(l = 1,2$ for $s = 1$ and $l = 2$ for $s = 2)$.

Then there exists a solution $u_{kl}(x,t)$ $(k = 1,2,\dots,m;\ l = 1,2)$ of problem (3)–(7) belonging to the class

$$H_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}(\overline{D_T^{(l)}}).$$

The proof of the existence of a solution of problem (3)–(7) is carried out by the classical method of continuation with respect to a parameter (see [7]), if one uses the $(2+\alpha)$-a priori estimate for the solution of problem (3)–(6), established by Theorem 4, and the existence of a solution from the class $H_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}(\overline{D_T^{(l)}})$ of the system (3^0), (4)–(7), established by Theorem 5 (equation (3^0) is obtained from (3) by putting in (3) $b_{ij}^{(kl)}(x,t) \equiv c_i^{(kl)}(x,t) \equiv 0$, $a_{ij}^{(kl)}(x,t) \equiv a_{ij}^{(kl)}$, where the matrix of constant coefficients $\|a_{ij}^{(kl)}\|$ is symmetric and positive definite).

Theorem 4. Suppose that all the conditions of Theorem 3 are satisfied (with 7) replaced by 6) from Theorem 2), and moreover

$$\det \|a_j^{(k)}(x,t)\| \ne 0, \quad (x,t) \in \Gamma^{(2)}, \qquad \det \|d_{jl}^{(k)}(x,t)\| \ne 0, \quad (x,t) \in \Gamma^{(1)}, \quad l = 1,2.$$

Suppose problem (3)–(6) has a solution $u_{kl}(x,t)$ for which

$$|u_{kl}|_{2+\alpha}^{D_T^{(l)}} < +\infty.$$

Then the estimate holds

$$|u_{kl}|_{2+\alpha}^{D_T^{(l)}} \leqslant C(D_T^{(l)}, d, d_0, \delta, a_0, M_0, M_1, M_2) \max_{i,j}\Big(\big|f_{ij}\big|_{\alpha}^{D_{T_j}{}^{(j)}} + \big|f_{ij}^{(1)}\big|_{2+\alpha}^{\Omega^{(j)}} + \big|f_j^{(2)}\big|_{1+\alpha}^{\Gamma^{(2)}} + \big|f_{ij}^{(3)}\big|_{1+\alpha}^{\Gamma^{(1)}} + \big|u_{ij}\big|_0^{D_T^{(j)}}\Big). \tag{9}$$

Remark. If uniqueness holds for problem (3)–(6) (see Theorems 1, 2), then, by virtue of the boundedness of D_T, in estimate (9) one may omit the term $\big|u_{ij}\big|_0^{D_T^{(j)}}$ appearing on the right.

Theorem 4 is proved by the same method by which, in the works of the author and V. N. Maslennikova ([8]) (see also ([2,4])), a $(2+\alpha)$-a priori estimate was established for the solution of a problem with an oblique derivative for a parabolic equation of second order in a noncylindrical domain.

Theorem 5. Suppose that for problem (3^0), (4)–(6), (8) all the conditions of Theorem 3 are satisfied. Then there exists a solution $u_{kl}(x,t)$ $(k = 1, 2, \dots, m;\ l = 1, 2)$ of problem (3^0), (4)–(7) belonging to the class

$$H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}(\overline{D_T^{(l)}}).$$

The proof of Theorem 5 is carried out by the method of potentials with the aid of a number of results (see Lemmas 1–6) of the theory of special Pani heat potentials ([9]), quite analogous to the theory of ordinary heat potentials from ([3,10,11]). If on $\Gamma \equiv \Gamma^{(2)}$, the lateral boundary of the domain D_T, a field of directions $\nu(x,t)$ satisfying (7) is prescribed, and $P(\bar{x},t;y,\tau)$ is the special fundamental solution of Pani ([9], §4), corresponding to $\nu(x,t)$ and to the parabolic operator with constant coefficients

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t},$$

then we consider the special heat surface potentials

potentials for $(\bar{x},t) \in D_T,\ (x,t) \in \Gamma$

$$P[\varphi] \equiv P(\bar{x},t) = \int_0^t d\tau \iint_{\Gamma_\tau} P(\bar{x},t;y,\tau)\varphi(y,\tau)\, d\sigma_y(\tau), \tag{10}$$

$$Q[\varphi] \equiv Q(\bar{x},t) = \frac{\partial P(\bar{x},t)}{\partial \nu(x,t)}$$

and their direct values on Γ: $\bar{P}[\varphi] = \bar{P}(x,t)$, $\bar{Q}[\varphi] = \bar{Q}(x,t)$, obtained from (10) for $(\bar{x},t) \equiv (x,t) \in \Gamma$.

Lemma 1. If Γ is of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$ $(0 < \alpha \leq 1)$, $[\varphi]_0^\Gamma < +\infty$, $[\nu]_0^\Gamma < +\infty$, then $P[\varphi] \in H^{0,\alpha_0,1/2}(\bar{D}_T)$ (α_0 is any number for which $0 < \alpha_0 < 1$), and the Hölder constants have the form $(C)[\varphi]_0^\Gamma$. Moreover, $P(\bar{x},0) \equiv 0$.

Lemma 2. If Γ is of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$ $(0 < \alpha \leq \beta \leq 1)$ and $[\varphi]_0^\Gamma < +\infty$, then $\bar{Q}[\varphi] \in H^{0,\alpha^*,\alpha^*/2}(\Gamma)$ ($\alpha^* = \min(\alpha,\beta')$, where β' is any number for which $0 < \beta' < \beta$), and the Hölder constants have the form $(C)[\varphi]_0^\Gamma$, and $\bar{Q}(x,0) \equiv 0$.

Lemma 3. Let Γ be of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$ $(0 < \alpha \leq \beta \leq 1)$, and let $(\bar{x},t) \in \Omega_t$ tend to x, $(x,t) \in \Gamma$, along the inner normal to Γ_t. Then for continuous φ

$$\lim_{(\bar{x},t)\to(x,t)\in\Gamma_t} Q(\bar{x},t) = \bar{Q}(x,t) - 2^{n-1}\pi^{n/2}(\det\|a^{ij}\|)^{-1/2}\varphi(x,t),$$

where $\|a^{ij}\|$ is the matrix inverse to $\|a_{ij}\|$. If $|\varphi|_\alpha^\Gamma < +\infty$ and $\varphi(x,0) \equiv 0$,

$$\left|Q(\bar{x},t) - \bar{Q}(x,t) + 2^{n-1}\pi^{n/2}(\det\|a^{ij}\|)^{-1/2}\varphi(x,t)\right| \leq (C)|\varphi|_\alpha^\Gamma|\bar{x}-x|^{\alpha^0}$$

$$(\alpha^0 = \alpha \text{ for } 0 < \alpha < 1;\ \alpha^0 = \alpha_0 \text{ for } \alpha = 1).$$

Lemma 4. Let Γ be of type ${}^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$, $|\nu|_{1+\beta}^\Gamma < +\infty$, $|\varphi|_\alpha^\Gamma < +\infty$ $(0 < \alpha \leq \beta \leq 1)$, and $\varphi(x,0) \equiv 0$. Then $Q[\varphi] \in H^{0,1,(1+\alpha^*)/2}_{1,\alpha',\alpha^*/2}(\Gamma)$, and the Hölder constants have the form $(C)|\varphi|_\alpha^\Gamma$, and $\bar{Q}(x,0) \equiv 0$.

Lemma 5. Let Γ be of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$, $|\varphi|_\alpha^\Gamma < +\infty$ $(0 < \alpha \leq \beta \leq 1)$, and $\varphi(x,0) \equiv 0$. Then $P[\varphi] \in H^{0,1,(1+\alpha')/2}_{1,\alpha',\alpha'/2}(\bar{D}_T)$, and the Hölder constants have the form $(C)|\varphi|_\alpha^\Gamma$, and $P(x,0) \equiv 0$.

Lemma 6. Let Γ be of type ${}^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$, $|\nu|_{1+\beta}^\Gamma < +\infty$, $|\varphi|_{1+\alpha}^\Gamma < +\infty$, $\varphi(x,0) \equiv 0$ $(0 < \alpha \leq \beta \leq 1)$. Then $P[\varphi] \in H^{1,\alpha',\alpha'/2}_{1,1,(1+\alpha')/2}(\bar{D}_T)$, and the Hölder constants have the form $(C)|\varphi|_{1+\alpha}^\Gamma$, and $P(x,0) \equiv 0$.

Remark. If Γ is of type ${}^{1,\beta,\beta/2}_{1,1,(1+\beta)/2}$, where $0 < \alpha < \beta \leq 1$, then, under the remaining assumptions of Lemma 6, $P[\varphi] \in H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}(\bar{D}_T)$.

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Note: Figure translations are in progress. See original paper for figures.