NONLINEAR EQUATIONS OF EQUILIBRIUM OF A SHELL

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Abstract

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V. M. DAREVSKII

NONLINEAR EQUATIONS OF EQUILIBRIUM OF A SHELL

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In investigating the stability of a shell, nonlinear equilibrium equations are used. They are formulated with allowance for the change in the middle surface of the shell under the action of the load. Usually only the rotations of the linear elements of the middle surface of the shell are taken into account ([1,2]). In some cases, for example for long cylindrical shells, it is also necessary to take elongations into account ([3]). The equilibrium equations of a shell of arbitrary shape, allowing for rotations and elongations of the linear elements of its middle surface, were derived in ([4]). Equations of this kind, but with allowance for shear, apparently have not been derived, although the possibility of neglecting shear in the general case has not been substantiated.

Below we give the general equilibrium equations of a shell that are obtained if rotations, elongations, and shear are taken into account. They apply to both principal cases: when the surface load is conservative and when it is follower.

Let the middle surface P of the undeformed shell be given by curvilinear coordinates α, β, where $\alpha = \text{const}, \beta = \text{const}$ are its lines of curvature; A^2, B^2 and R_1, R_2 are the coefficients of the first quadratic form and the principal radii of curvature.

It is assumed that on the surface P and on the deformed middle surface Π right-handed moving trihedra xyz and $\xi\eta\zeta$ are constructed, in which the axes x, y and ξ, η are directed along the tangents to the coordinate lines α ($\beta = \text{const}$), β ($\alpha = \text{const}$), respectively, on the surfaces P and Π in the direction of increasing parameters α, β, while the axes z and ζ are normal to these surfaces. At an arbitrary point of the surface Π the coordinate lines intersect at an angle χ, $\cos\chi = \omega$. The coordinates of the displacement vector of a point m of the surface P relative to the axes x, y, z with origin at the point m will be denoted by u, v, w. Let the point m of the surface P pass into the point μ of the surface Π. At the point m, the coordinates of the vector $\mathbf{q}$–the intensity of the load distributed over the surface (force referred to a unit area of the middle surface of the shell)–along the axes x, y, z with origin at the point m will be denoted by q_1, q_2, q_3. In the case of a conservative surface load, the quantities q_1, q_2, q_3

will also be the coordinates, relative to the indicated axes x, y, z, of the vector $\mathbf{q}$ at the point μ of the surface Π. If, however, the surface load is follower, then q_1, q_2, q_3 will be the coordinates of the vector $\mathbf{q}$ at the point μ of the surface Π relative to the axes ξ, η, ζ with origin at the point μ.

We single out an infinitely small element of the undeformed shell, whose middle surface ΔP is bounded by two pairs of coordinate lines α, β, of which one pair intersects at the point m. The lateral surfaces (ends) of this element are formed by the motion of a segment of the normal along the indicated coordinate lines α, β. Under the action of the load the selected element passes into an infinitely small element of the deformed shell with middle surface $\Delta\Pi$, bounded by two pairs of coordinate lines α, β on the surface Π, of which one pair intersects at the point μ. The coordinates along the axes ξ, η, ζ with vertex at the point μ of the force vector and moment vector acting at the point μ on

positive side of the cut along the line β (on the surface Π) be T_1, S_1, N_1 (the coordinates of the force) and $-M_{12}, M_1, 0$ (the coordinates of the moment). Let the analogous coordinates of the force vector and moment vector acting at the point μ on the positive side of the cut along the line α (on the surface Π) be, respectively, T_2, S_2, N_2 and $-M_2, M_{21}, 0$.

It seems expedient to write the equilibrium equations of an element of the deformed shell with middle surface $\Delta\Pi$ in the form of equations setting equal to zero the sums of like coordinates along the axes ξ, η, ζ of all internal forces and moments, as well as of the surface load, acting on the indicated element. Writing such equalities and passing to the limit as the side lengths of the element $\Delta\Pi$ tend to zero, we obtain the following equilibrium equations (with allowance for rotations and elongations of line elements and shear):

$$1.\ \frac{\partial}{\partial\alpha} T_1 B(1+\varepsilon_2) + \frac{\partial}{\partial\beta} S_2 A(1+\varepsilon_1) + S_1\left[\frac{\partial}{\partial\beta} A(1+\varepsilon_1) - \frac{\partial B}{\partial\alpha}\omega\right]$$
$$- T_2\left[\frac{\partial}{\partial\alpha} B(1+\varepsilon_2) - \frac{\partial}{\partial\beta} A\omega\right] + T_1\frac{\partial A}{\partial\beta}\omega - S_2\frac{\partial B}{\partial\alpha}\omega - N_1 AB\left(\frac{1}{R_1} + \chi_1 + \frac{\varepsilon_2}{R_1}\right)$$
$$- N_2 AB\left(\tau - \frac{\omega}{R_2}\right) + q_1 AB(1+\varepsilon_1+\varepsilon_2) + \binom{0}{1} AB(q_3\delta_1 - q_2\gamma_2) = 0.$$

$$2.\ \frac{\partial}{\partial\beta} T_2 A(1+\varepsilon_1) + \frac{\partial}{\partial\alpha} S_1 B(1+\varepsilon_2) + S_2\left[\frac{\partial}{\partial\alpha} B(1+\varepsilon_2) - \frac{\partial A}{\partial\beta}\omega\right]$$
$$- T_1\left[\frac{\partial}{\partial\beta} A(1+\varepsilon_1) - \frac{\partial}{\partial\alpha} B\omega\right] + T_2\frac{\partial B}{\partial\alpha}\omega - S_1\frac{\partial A}{\partial\beta}\omega - N_2 AB\left(\frac{1}{R_2} + \chi_2 + \frac{\varepsilon_1}{R_2}\right)$$
$$- N_1 AB\left(\tau - \frac{\omega}{R_1}\right) + q_2 AB(1+\varepsilon_1+\varepsilon_2) + \binom{0}{1} AB(q_3\delta_2 - q_1\gamma_1) = 0.$$

$$3.\ \frac{\partial}{\partial\alpha}N_1B(1+\varepsilon_2)+\frac{\partial}{\partial\beta}N_2A(1+\varepsilon_1)+T_1AB\left(\frac{1}{R_1}+\chi_1+\frac{\varepsilon_2}{R_1}\right)$$
$$+(S_1+S_2)AB\tau+T_2AB\left(\frac{1}{R_2}+\chi_2+\frac{\varepsilon_1}{R_2}\right)+q_3AB(1+\varepsilon_1+\varepsilon_2)$$
$$-\binom{0}{1}AB(q_1\delta_1+q_2\delta_2)=0.$$

$$4.\ \frac{\partial}{\partial\alpha}M_{12}B(1+\varepsilon_2)+\frac{\partial}{\partial\beta}M_2A(1+\varepsilon_1)-M_1\left[\frac{\partial}{\partial\beta}A(1+\varepsilon_1)-\frac{\partial B}{\partial\alpha}\omega\right]$$
$$+M_{21}\left[\frac{\partial}{\partial\alpha}B(1+\varepsilon_2)-\frac{\partial}{\partial\beta}A\omega\right]+M_{12}\frac{\partial A}{\partial\beta}\omega$$
$$-M_2\frac{\partial B}{\partial\alpha}\omega-N_2AB(1+\varepsilon_1+\varepsilon_2)=0.$$

$$5.\ \frac{\partial}{\partial\beta}M_{21}A(1+\varepsilon_1)+\frac{\partial}{\partial\alpha}M_1B(1+\varepsilon_2)-M_2\left[\frac{\partial}{\partial\alpha}B(1+\varepsilon_2)-\frac{\partial A}{\partial\beta}\omega\right]$$
$$+M_{12}\left[\frac{\partial}{\partial\beta}A(1+\varepsilon_1)-\frac{\partial}{\partial\alpha}B\omega\right]+M_{21}\frac{\partial B}{\partial\alpha}\omega$$
$$-M_1\frac{\partial A}{\partial\beta}\omega-N_1AB(1+\varepsilon_1+\varepsilon_2)=0.$$

$$6.\ (M_1-M_2)\tau-M_{12}\left(\frac{1}{R_1}+\chi_1+\frac{\varepsilon_2}{R_1}\right)+M_{21}\left(\frac{1}{R_2}+\chi_2+\frac{\varepsilon_1}{R_2}\right)$$
$$+(S_1-S_2)(1+\varepsilon_1+\varepsilon_2)=0.$$

In these equations the coefficients of the force factors are determined to within first powers (inclusive) of quantities of the order of the deformations; $\varepsilon_1,\varepsilon_2,\omega;\chi_1,\chi_2,\tau$ are the usual notations in shell theory for the components of the tangential and bending deformation of the middle surface of the shell (see (5), where the role of τ is played by τ^*), and

$$\tau=\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta}-\frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}-\frac{1}{BA^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha}+$$
$$+\frac{B}{AR_2}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)+\frac{A}{BR_1}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right).$$

or

$$\tau=\frac{1}{2}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\frac{\delta_1}{A}+\frac{B}{A}\frac{\partial}{\partial\alpha}\frac{\delta_2}{B}+\frac{\gamma_1}{R_2}+\frac{\gamma_2}{R_1}\right);$$

the quantities $\gamma_1,\gamma_2,\delta_1,\delta_2$ are determined by the equalities

$$\gamma_1 = \frac{1}{A}\left(\frac{\partial v}{\partial \alpha} - \frac{u}{B}\frac{\partial A}{\partial \beta}\right), \qquad \delta_1 = \frac{1}{A}\frac{\partial w}{\partial \alpha} + \frac{u}{R_1},$$

$$\gamma_2 = \frac{1}{B}\left(\frac{\partial u}{\partial \beta} - \frac{v}{A}\frac{\partial B}{\partial \alpha}\right), \qquad \delta_2 = \frac{1}{B}\frac{\partial w}{\partial \beta} + \frac{v}{R_2} \qquad (\gamma_1 + \gamma_2 = \omega);$$

the symbol $\binom{0}{1}$ denotes zero in the case of a follower surface load and unity in the case of a conservative load.

If the equations of equilibrium presented are used for investigating the stability of the shell and it is assumed, as is usually done, that the loss of stability is caused only by the action of the tangential forces T_1, T_2, S_1, S_2, then in these equations the deformation components in the coefficients at the moments and transverse forces N_1, N_2 may be discarded. Then the last three equations of equilibrium become the same as in the linear theory of shells and, in particular, the sixth equation reduces to an identity. In this case the system of equilibrium equations has the form:

$$\begin{aligned} 1.\quad & \frac{\partial}{\partial \alpha} T_1 B(1+\varepsilon_2) + \frac{\partial}{\partial \beta} S_2 A(1+\varepsilon_1) + S_1\left[\frac{\partial}{\partial \beta} A(1+\varepsilon_1) - \frac{\partial B}{\partial \alpha}\omega\right] \\ & - T_2\left[\frac{\partial}{\partial \alpha} B(1+\varepsilon_2) - \frac{\partial}{\partial \beta} A\omega\right] + T_1 \frac{\partial A}{\partial \beta}\omega - S_2 \frac{\partial B}{\partial \alpha}\omega - N_1 \frac{AB}{R_1} \\ & + q_1 AB(1+\varepsilon_1+\varepsilon_2) + \binom{0}{1} AB(q_3\delta_1 - q_2\gamma_2) = 0. \end{aligned}$$

$$\begin{aligned} 2.\quad & \frac{\partial}{\partial \beta} T_2 A(1+\varepsilon_1) + \frac{\partial}{\partial \alpha} S_1 B(1+\varepsilon_2) + S_2\left[\frac{\partial}{\partial \alpha} B(1+\varepsilon_2) - \frac{\partial A}{\partial \beta}\omega\right] \\ & - T_1\left[\frac{\partial}{\partial \beta} A(1+\varepsilon_1) - \frac{\partial}{\partial \alpha} B\omega\right] + T_2 \frac{\partial B}{\partial \alpha}\omega - S_1 \frac{\partial A}{\partial \beta}\omega - N_2 \frac{AB}{R_2} \\ & + q_2 AB(1+\varepsilon_1+\varepsilon_2) + \binom{0}{1} AB(q_3\delta_2 - q_1\gamma_1) = 0. \end{aligned}$$

$$\begin{aligned} 3.\quad & \frac{\partial}{\partial \alpha} N_1 B + \frac{\partial}{\partial \beta} N_2 A + T_1 AB\left(\frac{1}{R_1} + \chi_1 + \frac{\varepsilon_2}{R_1}\right) + (S_1 + S_2)AB\tau \\ & + T_2 AB\left(\frac{1}{R_2} + \chi_2 + \frac{\varepsilon_1}{R_2}\right) + q_3 AB(1+\varepsilon_1+\varepsilon_2) - \binom{0}{1} AB(q_1\delta_1 + q_2\delta_2) = 0. \end{aligned}$$

$$4.\quad \frac{\partial}{\partial \alpha} M_{12} B + \frac{\partial}{\partial \beta} M_2 A - M_1 \frac{\partial A}{\partial \beta} + M_{21} \frac{\partial B}{\partial \alpha} - N_2 AB = 0.$$

$$5.\quad \frac{\partial}{\partial \beta} M_{21} A + \frac{\partial}{\partial \alpha} M_1 B - M_2 \frac{\partial B}{\partial \alpha} + M_{12} \frac{\partial A}{\partial \beta} - N_1 AB = 0.$$

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Note: Figure translations are in progress. See original paper for figures.