RESONANCES IN MULTIFREQUENCY OSCILLATIONS

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Abstract

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MATHEMATICS

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RESONANCES IN MULTIFREQUENCY OSCILLATIONS

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It is shown that resonance in the general situation of a change of variables—the introduction of resonance phases—reduces to the special case of single-frequency resonance, when a frequency passes through zero. The concepts introduced are illustrated by an example of resonances in the Solar System.

§ 1. We study multifrequency oscillations of the form

$$\begin{aligned} \frac{dI}{dt} &= \varepsilon F(I, \varphi, \varepsilon), \\ \frac{d\varphi}{dt} &= \omega(I) + \varepsilon\Omega(I, \varphi, \varepsilon), \end{aligned} \tag{1}$$

Here ε is a small parameter; $I = (I_1, \dots, I_k)$ are slow variables; $\varphi = (\varphi_1, \dots, \varphi_l)$ are fast (phase variables). The right-hand sides are smooth functions of their arguments, periodic with period 2π in each phase.

The separation of the variables into fast and slow has an asymptotic meaning (as $\varepsilon \to 0$). If we put $\varepsilon = 0$, then the slow variables become first integrals of the system, $I = I_0$. The fast variables remain, generally speaking, variables even for $\varepsilon = 0$. However, some combinations of phases (which are commonly called resonant) may become constant for $\varepsilon = 0$. For a more precise description of the situation we introduce some definitions.

Definition 1. A surface in the space I of slow variables, defined by the equality

$$(\mathbf{n}, \vec{\omega}) \equiv n_1\omega_1(I) + \dots + n_l\omega_l(I) = 0, \tag{2}$$

where $\mathbf{n} = (n_1, \dots, n_l)$ is an integer vector, is called a **resonance surface**, and $\mathbf{n}$ the vector of the resonance.

Definition 2. An integer linear combination of phases

$$\psi = n_1\varphi_1 + \dots + n_l\varphi_l \tag{3}$$

is called the **resonance phase** of the given resonance.

Definition 3. The complexity index of a point I (or the complexity of the state I) is the number s of linearly independent resonance relations satisfied by the point I.

§ 2. In the space I of slow variables, the overwhelming majority of points—a set of full measure—are points of index 0, lying on none of the resonance surfaces (2). If one uses, as is often done, probabilistic terminology, one may say that the "probability for a point to have index 1 is zero." It is necessary, however, to take into account the slow motion, whose one-dimensional trajectories necessarily intersect resonance surfaces of index 1. Therefore, from the point of view of measure theory [1], it seems reasonable to study resonances of index 1. As for resonances of complexity 2, the "probability" of encountering such a point in a practical problem is zero.

All the more unexpected and remarkable is the result of the analysis of the resonance relations that occur in the Solar System. Anticipating somewhat, let us emphasize the main point. **In all the cases analyzed, the complexity has the maximum possible value.**

From what has been said above it is clear that this fact cannot be explained by chance. Another possible explanation seems much more satisfactory: the complete resonance of all subsystems of the Solar System is an inevitable result of long evolution.

3. Canonical form of a system of resonances. In what follows, when studying the phenomena that occur during passage through resonance, it is very important to be able to introduce, as independent phase variables, precisely the resonance phases of the given system of resonances. The replacement of phase variables

$$\psi = A\varphi, \qquad \varphi = B\psi \tag{4}$$

must preserve the form of system (1), i.e., periodicity in the phases. This means that both matrices A and B must have integer entries.

In the investigation, an essential role is played by the

Biorthogonalization theorem. *If the integer vectors $\mathbf{n}_1, \dots, \mathbf{n}_s$ are linearly independent, then there exists an integer biorthogonal system $\mathbf{a}_1, \dots, \mathbf{a}_s$; $\mathbf{b}_1, \dots, \mathbf{b}_s$ such that the vectors $\mathbf{n}_1, \dots, \mathbf{n}_s$ are obtained from the vectors $\mathbf{a}_1, \dots, \mathbf{a}_s$ by an integer triangular transformation*

$$\begin{aligned} \mathbf{n}_1 &= T_{11}\mathbf{a}_1, \\ \mathbf{n}_2 &= T_{21}\mathbf{a}_1 + T_{22}\mathbf{a}_2, \\ &\dots\dots\dots \\ \mathbf{n}_s &= T_{s1}\mathbf{a}_1 + T_{s2}\mathbf{a}_2 + \dots + T_{ss}\mathbf{a}_s. \end{aligned} \tag{5}$$

The theorem is proved by a method analogous to the well-known method of Lagrange orthogonalization (2), but with modifications resulting from the integrality of the problem. The chief one is the replacement of an orthonormal system by a biorthonormal one (2). In the proof one uses a lemma that follows easily from Euclid' s algorithm (3):

Lemma. *For any integer vector* $\mathbf{a}$ *there exists an integer vector* $\mathbf{b}$ *such that their scalar product* $d = (\mathbf{a}, \mathbf{b})$ *is equal to the greatest common divisor of the components of the vector* $\mathbf{a}$.

The theorem on reducing an arbitrary system of resonances

$$(\mathbf{n}_1, \vec{\omega}) = 0, \dots, (\mathbf{n}_s, \vec{\omega}) = 0 \tag{6}$$

to canonical form

$$\omega_1(I) = 0, \dots, \omega_s(I) = 0 \tag{7}$$

follows from the theorem formulated above almost immediately. For the proof, the resonance vectors $\mathbf{n}_1, \dots, \mathbf{n}_s$ must be completed in an arbitrary way to an integer basis $\mathbf{n}_1, \dots, \mathbf{n}_s, \mathbf{n}_{s+1}, \dots, \mathbf{n}_l$, and this basis biorthogonalized. The vectors $\mathbf{a}_i$, arranged in rows, form an integer square matrix A, and the columns $\mathbf{b}_k$ form a matrix B, with biorthogonality simply meaning that

$$AB = E. \tag{8}$$

Therefore the matrices A and B can be used to construct the change of variables (4). In this case the first s phases will automatically be resonant because of the triangularity of transformation (5).

4. Resonances in the Solar System. From the point of view of the theory of oscillations, any planetary system is a collection of weakly coupled oscillatory systems, the number of phases being equal to the number of planets. In the Solar System there are at least 4 such subsystems—9 planets, 4 satellites of Jupiter, 8 satellites of Saturn, and 5 satellites of Uranus.

Since the resonance relations correspond to the unperturbed problem ($\varepsilon = 0$), in computations for a concrete problem the vector $\mathbf{n}$ should be considered resonant if its scalar product with the frequency vector $\vec{\omega}$ is not equal to zero, but is of order ε: $(\mathbf{n}, \vec{\omega}) \sim \varepsilon$. For the Solar System, where $\varepsilon \sim 10^{-3}$ (the ratio of the masses of the planets to the mass of the Sun), this gives a value of the scalar product of several thousandths. This expectation is justified: the principal resonance of the Solar System—the $5:2$ resonance for the frequencies of Jupiter and Saturn—has an accuracy of about 1/2% (0.0067). Of the 22 resonance relations given below, only 3 have a lower accuracy; moreover, even the "worst" of them—the $1:2$ resonance for Neptune and Uranus—still has an accuracy of

$1\frac{1}{2}\%$. In the frequency table (4), the frequency of the most massive body in each system is taken as unity.

Frequency Table

Planets		Satellites of Jupiter		Satellites of Saturn	
Mercury	49.22	Io	4.044	Mimas	16.918
Venus	19.29	Europa	2.015	Enceladus	11.639
Earth	11.862	Ganymede	1.000	Tethys	8.448
Mars	6.306	Callisto	0.4288	Dione	5.826
Jupiter	1.000	Satellites of Uranus		Rhea	3.530
Saturn	0.4027	Miranda	6.529	Titan	1.000
Uranus	0.14119	Ariel	3.454	Hyperion	0.7494
Neptune	0.07197	Umbriel	2.100	Iapetus	0.2010
Pluto	0.04750	Titania	1.000		
		Oberon	0.6466		

Table of Resonance Vectors

Planets	Satellites of Saturn
(1 1 2 1 0 0 0 0 0)	(1 0 −2 0 0 0 0 0)
(−0 −1 0 3 0 1 0 0 0)	(0 −1 0 2 0 0 0 0)
(0 0 −1 2 −1 1 −1 0 0)	(0 0 −1 0 2 1 0 2)
(0 0 0 1 −6 0 −2 0 0)	(0 0 0 −1 2 −1 0 −1)
(0 0 0 0 −2 5 0 0 0)	(0 0 0 0 1 −2 −2 0)
(0 0 0 0 −1 0 7 0 0)	(0 0 0 0 0 3 −4 0)
(0 0 0 0 0 0 −1 2 0)	(0 0 0 0 0 −1 0 5)
(0 0 0 0 0 0 −1 0 3)	

Satellites of Jupiter	Satellites of Uranus
(1 −2 0 0)	(−1 1 1 1 0)
(0 1 −2 0)	(0 1 −1 −2 1)
(0 −3 0 7)	(0 0 −2 1 5)
	(0 0 1 −4 3)

Analysis of the resonance tables leads to the following conclusions:

1. The rule of maximum resonance is applicable to all systems of satellites and to the system of planets—the number of resonance relations is exactly one less than the number of phases.

2. Systems with a small number of members are fairly homogeneous—the resonance relations encompass the majority of the participants.
3. The system of planets and the satellites of Saturn (consisting of 9 and 8 members, respectively) already reveal a clear tendency toward the formation of a heterogeneous structure.

Indeed, the system of planets naturally breaks up into three groups: Mercury—Venus—Earth—Mars, Jupiter—Saturn, and Uranus—Neptune—Pluto. Within each group there are unifying resonances: the first, second, and third for the Earth group, the fifth for the Jupiter group, the seventh and eighth for the Uranus group; at the same time the subordinate role of the terrestrial-planet group in relation to the other groups is clearly visible. A special role is played by the fourth and sixth resonances, which unite the three coalitions into a single Solar System.

The system of Saturn's satellites reveals a similar structure, but the coalitions are more equal in status. The coalitions themselves are: Mimas—Tethys, Enceladus—Dione, Rhea—Titan—Hyperion—Iapetus. In the table it is easy to find the "local" and "general" resonances.

5. On the law of planetary distances. Let us note an important property of states with the maximal complexity index. States with the maximal complexity index are uniquely specified by the table of resonances. Indeed, in this case all frequencies can be expressed in terms of one, which remains free. A change in the free frequency corresponds simply to a change in the scale of the system.

Specifically for planetary systems this means that the formulation of the question of a law of planetary distances (5) is unsuccessful, since simple integer relations are obtained not for distances, but for frequencies, through which the distances are determined uniquely. Moreover, the law of planetary distances, even if applicable, is applicable only to planets, whereas the rule of maximal resonance complexity is equally well suited to all the cases considered.

It is impossible to refrain from observing that integrality, usually associated with quantum physics, is probably a general property of sufficiently old systems. Quantum systems are simply always old for us, since their time scale is usually negligible, and we "find" them already evolved. It is possible that it was precisely for this reason that integrality first attracted attention in the physics of elementary particles, and even determined its name—quantum.

The main idea may be formulated briefly as follows: the resonance of a system is a consequence (and a sign) of its evolutionary maturity.

This point of view finds curious confirmation in a circumstance usually noted only as an amusing accident. It is known ([6]) that motion in an arbitrary centrally symmetric field is, generally speaking, two-frequency, and the angular and radial frequencies are in no way connected with one another. But precisely in the case of the Newtonian potential there is an identical (for all values of angular momentum and energy) resonance of these frequencies in the ratio $1 : 1$.

Consequently, the motion of even a single planet already satisfies the rule of maximal resonance. This fact is very satisfactory from the evolutionary point of view, since the Newtonian potential corresponds to the interactions with the largest time scales—the electric and the gravitational.

Another identically resonant potential is known—the potential of the harmonic oscillator, with resonance $1:2$. It would be interesting to prove that no other identically resonant potentials exist.

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Note: Figure translations are in progress. See original paper for figures.