EINSTEIN' S EQUATIONS AS EQUATIONS FOR A MASSLESS TENSOR FIELD WITH SPIN 2

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Abstract

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V. I. OGIEVETSKII, I. V. POLUBARINOV

EINSTEIN' S EQUATIONS AS EQUATIONS FOR A MASSLESS TENSOR FIELD WITH SPIN 2

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1. In the theory of interactions of fields with higher spins (see, for example, ([1])) there are unexplored possibilities which may lead to unexpected results. Thus, the use of the concept of the spin of an interacting field ([2]) and of the principle of restricting fields by spin leads to a narrowing of the class of interactions of vector fields with spin 1 and to far-reaching consequences following from this ([3,4]). In the case of one massless vector field, Maxwell electrodynamics is obtained unambiguously; in the case of three vector fields with spin 1—isotopic invariance and the Yang-Mills theory, etc. What will the theory of an interacting tensor field $h^{\mu\nu}$ with spin 2 be? In the present note it is proved that the equations of motion of such a theory coincide with Einstein' s equations ([5]) for the gravitational field. Thus a field-theoretic derivation is given of Einstein' s equations with all their nonlinearities in flat space as equations for an interacting tensor field with spin 2.

2. An interacting symmetric tensor field $h^{\mu\nu}$ ($h^{\nu\mu} = h^{\mu\nu}$) with mass m, satisfying the additional condition*

$$\partial_\mu h^{\mu\nu} + q\partial_\nu h^{\mu\mu} = \begin{cases} 0, & \text{for } m \neq 0; \\ \text{arbitrary for } m = 0 & \end{cases} \tag{1a}$$

(1)

(q is an arbitrary number), describes spins 2 and 0, while spin 1 and another spin 0 are excluded. Spin 1 must necessarily be excluded, since the sign of the energy for spin 1 is opposite to the sign of the energy for spins 2 and 0. Let us note that the field $h^{\mu\nu}$ would describe only spin 2 if condition (1) were supplemented by a second condition: $h^{\mu\mu}$ equal to zero or arbitrary. The analysis of theories in which both additional conditions are fulfilled is greatly complicated by the fact that they are not completely independent.

In the present paper the task is set of enumerating theories of minimal interactions of the tensor field in which one condition—condition (1)—is fulfilled, and it is proved that in the case of a massless tensor field there exists only one such theory, the equations of motion in which coincide with Einstein' s equations. We call interactions minimal if they contain nonzero coupling constants of dimension cm and only those coupling constants of other dimensions whose being set equal to zero would lead to the vanishing of all coupling constants of dimension cm (i.e., to the complete switching off of the interaction). In the case of vector fields, minimality naturally meant dimensionless coupling constants, whereas in theories of tensor fields the essential coupling constants are precisely of dimension cm. It is methodologically more convenient for us to start from a massive tensor field and to require the fulfillment of the condition

* The choice of an additional condition for a massless tensor field in the form $\partial_\mu h^{\mu\nu} = 0$ (fixing the arbitrariness– "gauge") corresponds to the de Donder-Lanczos-Fock condition, well known and widely used by Fock in the theory of gravitation.

of Hilbert-Lorentz type (1a), although in the present article we shall ultimately be interested in the field with zero mass.

3. Let us consider the interaction of a tensor field with itself and with a scalar field φ: $\mathcal{L} = \mathcal{L}(h^{\mu\nu}, \partial_\lambda h^{\mu\nu}, \partial_\lambda \partial_\rho h^{\mu\nu}, \varphi, \partial_\lambda \varphi)$, where, for convenience, second derivatives of $h^{\mu\nu}$ have been introduced. Let us represent the Lagrangian density $\mathcal{L}$ in the form

$$\mathcal{L} = \mathcal{L}' - {}^1/_2 m^2 (h^{\mu\nu} h^{\mu\nu} + q h^{\mu\mu} h^{\nu\nu}). \tag{2}$$

In separating out this mass term, we do not assume that $\mathcal{L}'$ is independent of m. Obtaining from (2) the equation of motion for the tensor field and taking its 4-divergence, we find

$$m^2 (\partial_\mu h^{\mu\nu} + q \partial_\nu h^{\rho\rho}) = \partial_\mu \left(\frac{\delta \mathcal{L}'}{\delta h^{\mu\nu}} - \partial_\lambda \frac{\delta \mathcal{L}'}{\delta \partial_\lambda h^{\mu\nu}} + \partial_\lambda \partial_\rho \frac{\delta \mathcal{L}'}{\delta \partial_\lambda \partial_\rho h^{\mu\nu}} \right). \tag{3}$$

It will follow from the equations of motion that condition (1) holds if the right-hand side of (3) vanishes either identically or as a consequence of the equations of motion. Therefore, for the Lagrangian density the identity

$$\partial_\mu \left(\frac{\delta \mathcal{L}'}{\delta h^{\mu\nu}} - \partial_\lambda \frac{\delta \mathcal{L}'}{\delta \partial_\lambda h^{\mu\nu}} + \partial_\lambda \partial_\rho \frac{\delta \mathcal{L}'}{\delta \partial_\lambda \partial_\rho h^{\mu\nu}} \right) \equiv -\frac{1}{2} \mathfrak{D}^\nu \left(\frac{\delta \mathcal{L}'}{\delta \varphi} - \partial_\lambda \frac{\delta \mathcal{L}'}{\delta \partial_\lambda \varphi} \right) - \frac{1}{2} \mathfrak{A}^\nu_{\alpha\beta} \left[\frac{\delta \mathcal{L}'}{\delta h^{\alpha\beta}} - \partial_\lambda \frac{\delta \mathcal{L}'}{\delta \partial_\lambda h^{\alpha\beta}} + \partial_\lambda \partial_\rho \frac{\delta \mathcal{L}'}{\delta \partial_\lambda \partial_\rho h^{\alpha\beta}} - m^2 (h^{\alpha\beta} + q \delta^{\alpha\beta} h \right. \tag{4}$$

where $\mathfrak{A}$ and $\mathfrak{D}$ are certain operators acting on the Eulerians, must hold. Within the framework of the principle of minimality we shall assume that they do not contain constants with dimensions higher than cm. Then

$$\mathfrak{A}^{\nu}_{\alpha\beta} = p\delta^{\alpha\beta}\partial_\nu + A^{\nu\varepsilon}_{\alpha\beta,\gamma\delta}h^{\gamma\delta}\partial_\varepsilon + \overline{A}^{\nu\varepsilon}_{\alpha\beta,\gamma\delta}\partial_\varepsilon h^{\gamma\delta}, \qquad \mathfrak{D}^{\nu} = D^{\nu\varepsilon}\varphi\partial_\varepsilon + \overline{D}^{\nu\varepsilon}\partial_\varepsilon\varphi, \quad (5)$$

where p is a dimensionless constant; $A, \overline{A}, D$, and $\overline{D}$ are numerical matrices independent of x, whose elements have the dimension cm. The coefficient $-{}^1/_2$ has been introduced for convenience.

4. We shall vary $\mathcal{L}$ on the special class of variations*

$$\delta^* h^{\mu\nu} = \partial_\mu\lambda^\nu + \partial_\nu\lambda^\mu + p\delta^{\mu\nu}\partial_\sigma\lambda^\sigma + \left(A^{\rho\varepsilon}_{\mu\nu,\sigma\tau}\partial_\varepsilon\lambda^\rho + \overline{\overline{A}}^{\rho\varepsilon}_{\mu\nu,\sigma\tau}\lambda^\rho\partial_\varepsilon\right)h^{\sigma\tau}, \quad (6)$$

$$\delta^*\varphi = \left(D^{\rho\varepsilon}\partial_\varepsilon\lambda^\rho + \overline{\overline{D}}^{\rho\varepsilon}\lambda^\rho\partial_\varepsilon\right)\varphi, \quad (7)$$

where $\lambda^\rho(x)$ is an arbitrary 4-vector function, $\overline{\overline{A}} = A - \overline{A}$, $\overline{\overline{D}} = D - \overline{D}$. Then identity (4) is written in the language of the variational principle as

$$\int d^4x\,\delta^*\mathcal{L} = -2m^2\int d^4x\left(h^{\mu\nu} + q\delta^{\mu\nu}h^{\rho\rho}\right)\partial_\mu\lambda^\nu. \quad (8)$$

Identity (8), for $m \neq 0$, in an obvious way guarantees the fulfillment of the supplementary condition (1). In the limit $m = 0$

$$\int d^4x\,\delta^*\mathcal{L} = 0, \quad (9)$$

i.e. for a massless field identity (8) becomes the condition of invariance of the theory with respect to the transformations (6) and (7) with arbitrary λ^μ. In this case the transformations (6), (7) turn out to be gauge transformations intended to eliminate spin 1 from the massless tensor field—an analogue of the gauge transformations in electrodynamics and Yang-Mills theories.

5. Identity (4) is in fact four identities ($\nu = 1, 2, 3, 4$), which are sufficient for eliminating four degrees of freedom (spins 1 and 0). It is natural to suppose that there are no other identities restricting

* We use Pauli' s notation δ^* in order to emphasize that the variations are local, i.e. do not affect the coordinates.

degrees of freedom of the tensor field. Identity (4) for $m = 0$ is an identity of the type that appears in E. Noether' s second theorem. By Noether' s theorem, from the existence of identities one may again arrive at the conclusion of invariance, and from the fact that there are four identities and only four it follows that transformations (6), (7) must form a group.

6. Consequently, transformations (6), (7) must satisfy the group structure relations

$$(\delta^*_{\lambda_2}\delta^*_{\lambda_1} - \delta^*_{\lambda_1}\delta^*_{\lambda_2})h^{\mu\nu} = \delta^*_{\lambda_{\text{ck}}}h^{\mu\nu}, \qquad (\delta^*_{\lambda_2}\delta^*_{\lambda_1} - \delta^*_{\lambda_1}\delta^*_{\lambda_2})\varphi = \delta^*_{\lambda_{\text{ck}}}\varphi, \tag{10}$$

where $\delta^*_{\lambda_1}$, $\delta^*_{\lambda_2}$, and $\delta^*_{\lambda_{\text{ck}}}$ are variations (6) or (7) with the function λ_1^ρ, with some other function λ_2^ρ, and with the "commutator" function λ_{ck}^ρ, constructed from λ_1^ρ and λ_2^ρ. Solving the structure relations (10), we find that all their solutions reduce, by a change of field variables, to

$$\delta^* h_p^{\mu\nu} = \partial_\mu\lambda^\nu + \partial_\nu\lambda^\mu + p\delta^{\mu\nu}\partial_\sigma\lambda^\sigma +$$

$$+a(\partial_\sigma\lambda^\nu h_p^{\mu\sigma} + \partial_\sigma\lambda^\mu h_p^{\sigma\nu} + p\partial_\sigma\lambda^\sigma h_p^{\mu\nu} - \lambda^\sigma\partial_\sigma h_p^{\mu\nu}), \tag{11}$$

$$\delta^*\varphi_s = -a\lambda^\rho\partial_\rho\varphi_s + s\partial_\rho\lambda^\rho\varphi_s, \tag{12}$$

where $\lambda_{\text{ck}}^\mu = -a(\lambda_1^\rho\partial_\rho\lambda_2^\mu - \lambda_2^\rho\partial_\rho\lambda_1^\mu)$, and a and s are dimensional constants. The transition from $h_p^{\mu\nu}$ to the quantity $g_p^{\mu\nu} = \delta^{\mu\nu} + ah_p^{\mu\nu}$ suggests itself; for this quantity

$$\delta^* g_p^{\mu\nu} = a(\partial_\sigma\lambda^\nu g_p^{\mu\sigma} + \partial_\sigma\lambda^\mu g_p^{\sigma\nu} + p\partial_\sigma\lambda^\sigma g_p^{\mu\nu} - \lambda^\sigma\partial_\sigma g_p^{\mu\nu}). \tag{13}$$

It is noteworthy that the laws of gauge transformations of the scalar and tensor fields have coincided with the laws of infinitesimal local transformations of a scalar (with weight s) and a contravariant tensor of second rank (with weight p) in Riemannian geometry (5). Hence, regarding Riemannian geometry as a set of rules for constructing invariants with respect to transformations (11), (12), we arrive at the conclusion that the equations of motion of the theory under investigation must coincide exactly with the equations of motion of Einstein' s theory of gravitation.

7. The corresponding Lagrangian can, closing one' s eyes to Riemannian geometry, also be reconstructed directly from the identity

$$\delta^*\mathcal{L} \equiv \partial_\mu X^\mu, \tag{14}$$

which follows from (9). For compatibility with Lorentz invariance, X^μ should be chosen in the form $X^\mu = -a\lambda^\mu\mathcal{L}$. Then the general solution of identity (14) will be a scalar (with weight $s = -1$) density $\mathcal{L} = \mathcal{L}(g_p^{\mu\nu}, R^{\rho}_{\mu\nu\lambda}, \varphi, \partial_\mu \varphi)$, where $R^{\rho}_{\mu\nu\lambda}$ is the "curvature tensor" with one raised index. Requiring that the equations be no higher than second order and that in the limit $a \to 0$ the Lagrangian pass into the sum of the free Lagrangians for the fields $h^{\mu\nu}$ and φ, we obtain the Einstein Lagrangian

$$\mathcal{L} = \left(-\frac{2}{a^2}R - \frac{1}{2}g^{\lambda\rho}\partial_\lambda\varphi\partial_\rho\varphi - \frac{\mu^2}{2}\varphi\varphi\right) g_0^{-1/2}, \tag{15}$$

where $g_0 = \mathrm{Det}\,\|g^{\mu\nu}\|$, R is the scalar "curvature," expressed through $g^{\mu\nu}$ in the usual way, and the fields $g^{\mu\nu}$ and φ used here are assumed to have zero weights ($p = s = 0$). One can always pass to zero weights by means of the transformations $g_p^{\mu\nu} = g_0^{p/2} g^{\mu\nu}$, $\varphi_s = g_0^{s/2a}\varphi$. From (15) follow the Einstein equations of motion

$$R_{\mu\nu} - {}^1/_2 g_{\mu\nu} R = -{}^1/_{4a^2} T_{\mu\nu} \qquad \left(T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - {}^1/_2 g_{\mu\nu}(g^{\lambda\rho}\partial_\lambda\varphi\partial_\rho\varphi + \mu^2\varphi\varphi)\right), \tag{16}$$

where

$$g_{\mu\nu} = \delta^{\mu\nu} - ah^{\mu\nu} + a^2 h^{\mu\lambda}h^{\lambda\nu} - \ldots \quad \text{(geometric progression)} \tag{17}$$

and we arrive at the conclusion that the coupling constant a used by us is related to Einstein' s constant χ by the relation $a^2 = 4\chi$. The Lagrangian (15) and equation (16) are, taking (17) into account, infinite series in the gravitational field $h^{\mu\nu}$, in accordance with Gupta' s interpretation ([8]). The inclusion of interactions with vector and spinor fields also presents no difficulty ([6]).

8. Thus, a field-theoretic derivation of the Einstein equations in flat space-time has been given on the basis of the spin principle.* Attempts to interpret the Einstein equations as nonlinear equations in flat space-time have a rather long history (see, for example, ([8,13,14])). Until now, however, there remained dissatisfaction with the inability to derive the Einstein equations in flat space ([13]). In the present article we fill this gap.

Despite the difference in the interpretation of space-time in the field-theoretic and geometrical approaches, both give identical predictions for observable effects (see ([13,14])). Thus, the old approach, using the concepts of conservation laws, forces, and fields (including the gravitational one), proves acceptable in the theory of gravitation, alongside Einstein' s original approach, in which gravitation is reduced to the curvature of space-time. In the field approach, the widespread tendency to make all fields equal in rights is realized by reducing the gravitational field to the rank of ordinary fields, and not by geometrizing all fields in addition to the gravitational one. Such an approach is very fruitful for quantization and for calculating effects in perturbation theory ([8,15]).

Let us note that, in accordance with the fact that spin 0 was also left in the interacting tensor field, virtual gravitons carry both spin values, spin 2 and spin 0. This was directly clarified in the weak-field approximation by V. I. Zakharov ([16]). At the same time, it is well known that emitted gravitons have pure spin 2.

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* Let us note that recently the Einstein equations have been interpreted in the spirit of the ideas of Yang-Mills ([9–11]). The principal difference of our approach is that: 1) our transformation group is not postulated, but is derived as a means of restricting the tensor field by spin, and we consistently adhere to flat space-time; 2) in constructing the theory, we have no ambiguity in the choice of fields, since we study the interaction of precisely a tensor field. Weinberg' s paper ([12]) has just appeared, in which an attempt is made to derive the Einstein equations within the framework of the S-matrix formalism and perturbation theory; however, the author does not go beyond the weak-field approximation.

Note: Figure translations are in progress. See original paper for figures.