ON THE SOLVABILITY OF THE BASIC BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF PARABOLIC TYPE WITH DEGENERATING PRINCIPAL PART

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Abstract

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ON THE SOLVABILITY OF THE BASIC BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF PARABOLIC TYPE WITH DEGENERATING PRINCIPAL PART

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Let $\Omega = G \times [0, t_0]$ be a cylinder in the Euclidean space of variables x, y, t. Suppose that the boundary Γ of the plane domain G consists of the segment $\delta: y = 0,\ a \leq x \leq b$, and a Jordan curve σ with endpoints at the points $(a, 0)$ and $(b, 0)$, lying in the half-plane $y > 0$. We shall say that the domain G belongs to the class A_1 if its boundary Γ has everywhere nonnegative piecewise-continuous curvature. Denote by $W^{2,1}_{2(y)}(\Omega)$ the space of functions $u(x, y, t)$ summable in the domain Ω, having generalized derivatives of first order with respect to t and of first and second orders with respect to x and y, with finite norm

$$\|u\|^2_{W^{2,1}_{2(y)}} = \int_\Omega \left(y^2 u^2_{xx} + y u^2_{xy} + u^2_{yy} + y u^2_x + u^2_y + u^2_t + u^2\right) d\Omega. \qquad (1)$$

Similarly, we introduce the space $W^2_{2(y)}(\Omega)$ of functions $u(x, y)$, $(x, y) \in G$, with finite norm

$$\|u\|^2_{W^2_{2(y)}} = \int_G \left(y^2 u^2_{xx} + y u^2_{xy} + u^2_{yy} + y u^2_x + u^2_y + u^2\right) dG. \qquad (2)$$

The spaces $W^2_{2(y)}$ and $W^{2,1}_{2(y)}$ are complete. Indeed, owing to the boundedness of the domain G, the inequality

$$\|u\|^2_{W^2_{2(y,2,2,2)}} = \int_G y^2 \left(u^2_{xx} + u^2_{xy} + u^2_{yy} + u^2_x + u^2_y + u^2\right) dG \leq c\|u\|^2_{W^2_{2(y)}}, \qquad (3)$$

holds, where $W^2_{2(y,2,2,2)}$ is the complete space ([1]).

By virtue of (3), every fundamental sequence $\{u_n\}$ in $W^2_{2(y)}$ will converge to a function $v(x,y)$ having generalized derivatives up to second order, and in order to prove completeness of $W^2_{2(y)}(G)$ it is enough to show that $v(x,y) \in W^2_{2(y)}(G)$.

By the fundamentalness of the sequence $\{u_n\}$ in $W^2_{2(y)}$, the inequality

$$\int_G y\left[(u_n - u_k)_{xy}\right]^2 dG < \varepsilon,$$

holds, i.e.

$$\int_G y(u_{nxy} - u_{kxy})^2 \, dG < \varepsilon.$$

Hence it follows that $\{u_{nxy}\}$ is fundamental in $W^0_{2(y,1)}$. Owing to the completeness of $W^0_{2(y,1)}$, there exists a function $\omega \in W^0_{2(y,1)}$ such that

$$\int_G y(u_{nxy} - \omega)^2 \, dG < \varepsilon,$$

but then also

$$\int_G y^2(u_{nxy} - \omega)^2 \, dG < \varepsilon,$$

i.e. $\{u_{nxy}\}$ converges to ω in $W^0_{2(y,2)}$. Taking into account the convergence of $\{u_{nxy}\}$ to v_{xy} in the space $W^0_{2(y,2)}$,

we conclude that v_{xy} coincides with ω almost everywhere. Thus,

$$\int_G y v_{xy}^2 \, dG$$

exists.

The existence of the other integrals on the right-hand side of (2) in the expression $\|v\|_{W^2_{2(y)}}$ is proved analogously.

Now note that functions from $W^2_{2(y)}$ belong to the space $W^1_{2(y,1,0)}\Omega$, and, moreover, in each section $t = \text{const}$ of the domain Ω they may be regarded as functions of the space $W^2_{2(y)}$. Repeating the reasoning given above, one easily verifies the validity of the second part of our assertion.

The main boundary-value problem. Determine in Ω a solution of the equation

$$Lu \equiv Tu - u_t \equiv y u_{xx} + u_{yy} - u_t = f(x,y,t) \tag{4}$$

under the condition

$$u\big|_{G\cup S} = 0, \tag{5}$$

where S is the lateral surface of the cylinder Ω.

For an equation of parabolic type in the absence of degeneration of the principal part, the problem with boundary condition (5) in the class of functions $u \in W_2^2(\Omega)$ with respect to x, y and $W_2^1(\Omega)$ with respect to t was solved in ([2]). For degenerating parabolic equations with a sufficiently smooth right-hand side in domains with smooth boundary, an analogous problem was considered in ([3]).

Denote by $W^2_{2,0(y)}$ the space of functions $u(x,y)$ obtained by completing, in the norm $W^2_{2(y)}$, the functions three times continuously differentiable in $\overline{G}$ and vanishing on Γ. On the basis of the obvious inequality

$$\|u\|_{W_2^2(G_\varepsilon)} \le c\|u\|_{W^2_{2(y)}(G)},$$

where $G_\varepsilon \subset G$ is the domain obtained by moving a distance $\varepsilon > 0$ away from the x-axis, by the Sobolev embedding theorem ([4]), the functions $u(x,y) \in W^2_{2,0(y)}(G)$ are continuous in $\overline{G_\varepsilon}$ and vanish on $\sigma \cap \overline{G_\varepsilon}$.

The inequality ([5]) holds:

$$\int_\delta u^2\, dx \le c \int_G (yu_x^2 + u_y^2 + u^2)\, dG$$

for functions vanishing on Γ, from which it follows that functions from $W^2_{2,0(y)}(G)$ vanish on the segment δ in the mean.

The space $W^{2,1}_{2,0(y)}$ of functions $u(x,y,t)$ is introduced analogously. Functions $u \in W^{2,1}_{2,0(y)}$ also vanish on $(G \cup S)$ in the mean. This follows from the fact that, for fixed t, they may be regarded as functions from $W^2_{2,0(y)}$, and, moreover,

$$\int_G t_0 u^2(x,y,t)\, dG = \int_\Omega [u^2 - 2(t_0 - t)uu_t]\, d\Omega \le c\|u\|_{W^{2,1}_{2(y)}\Omega}.$$

Let us first consider the homogeneous Dirichlet problem for the equation $Tu = h(x,y)$ in the domain G.

Lemma 1. If $u \in W^2_{2,0(y)}(G)$, then

$$\|Tu\|_{L_2(G)} \ge c\|u\|_{W^2_{2(y)}(G)}. \tag{6}$$

It is sufficient to carry out the proof for functions having continuous derivatives up to the third order in $\overline{G}$ and vanishing on Γ, and then, by means of a limiting passage, to extend it to all functions from $W^2_{2,0(y)}(G)$.

Using Hölder' s inequality, from Green' s formula, by virtue of the condition

$$u|_{\Gamma} = 0, \tag{7}$$

we obtain

$$c_1 \int_G u^2 \, dG \leq c \int_G (yu_x^2 + u_y^2) \, dG \leq \left[\int_G u^2 \, dG \right]^{1/2} \left[\int_G (yu_{xx} + u_{yy})^2 \, dG \right]^{1/2}, \tag{8}$$

whence

$$\int_G (yu_x^2 + u_y^2 + u^2) \, dG \leq c \int_G (Tu)^2 \, dG.$$

Let $x = x(s)$, $y = y(s)$ $(0 \leq s \leq s_0, \ y(0) = 0, \ x(0) = b, \ y(s_0) = 0, \ x(s_0) = a)$ be the parametric equation of σ. We divide the curve σ into a finite number of arcs σ_i in such a way that on $\sigma_1, \sigma_2, \dots, \sigma_k$ the inequality $x_s' \neq 0$ holds, and on $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{k+r}$ the inequality $y_s' \neq 0$ holds. In view of the fact that, by (7), $du/ds|_\sigma = d^2u/ds^2|_\sigma = 0$, on the basis of the obvious identity

$$2 \int_G yu_{xx}u_{yy} \, dG = \int_\sigma (yu_{xu_{yy}} - yu_{yu_{xy}} - u_{xu}y) \, dy +$$

$$+ (yu_{xu_{xy}} - yu_{yu_{xx}} - u_x^2) \, dx + 2 \int_G yu_{xy}^2 \, dG,$$

we shall have

$$2 \int_G yu_{xx}u_{yy} \, dG = \int_{\sigma_i} \sum_{i=1}^{k} \frac{y(x'y'' - x''y')}{(x')^2} \, u_y^2 \, ds +$$

$$+ \int_{\sigma_i} \sum_{i=k+1}^{k+r} \frac{y(x'y'' - x''y')}{(y')^2} \, u_x^2 \, ds + 2 \int_G yu_{xy}^2 \, dG. \tag{9}$$

As a consequence of the nonnegativity of the curvature of σ_i, from (9) we conclude

$$\int_G yu_{xy}^2 \, dG \leq \int_G yu_{xx}u_{yy} \, dG, \tag{10}$$

whence the validity of Lemma 1 follows.

We shall say that the domain G belongs to the class A_2 if the homogeneous Dirichlet problem for the Poisson equation $\Delta u = h(x,y)$ is solvable in $W^2_{2,0}(G)$ (5) for every right-hand side h from $L_2(G)$.

Theorem 1. If the domain G belongs to $A_1 \cap A_2$, then the homogeneous Dirichlet problem for the equation

$$Tu = f(x,y) \tag{11}$$

is uniquely solvable in $W^2_{2,0(y)}$ for any $f \in L_2(G)$.

The range of the operator T, defined on functions from $W^2_{2,0(y)}$, by Lemma 1, is closed in L_2. Consequently, to prove Theorem 1 it remains to show that the following is true.

Lemma 2. If $v \in L_2(G)$ and

$$\int_G vTu\, dG = 0 \tag{12}$$

for all $u \in W^2_{2,0(y)}$, then $v \equiv 0$.

Since the domain G belongs to the class A_2, there exists a function $\bar{u} \in W^2_{2,0}(G) \subset W^2_{2,0(y)}(G)$ such that $\Delta\bar{u} = v$. Substituting in (12), instead of u and v, respectively $u = \bar{u},\ v = \Delta\bar{u}$, we obtain

$$\int_G \Delta\bar{u}\, T\bar{u}\, dG = \int_G \left[y\bar{u}^2_{xx} + \bar{u}^2_{yy} + (1+y)\bar{u}_{xx}\bar{u}_{yy}\right] dG.$$

For the function $\bar{u}$ there is an inequality analogous to (10):

$$\int_G (1+y)\bar{u}_{xx}\bar{u}_{yy}\, dG \geqslant \int_G (1+y)\bar{u}^2_{xy}\, dG,$$

therefore,

$$c\|\bar{u}\|_{W^2_{2(y)}} \leqslant c\int_G (T\bar{u})^2\, dG \leqslant \int_G \left(y\bar{u}^2_{xx} + \bar{u}^2_{yy} + \bar{u}^2_{xy}\right) dG \leqslant \int_G \Delta\bar{u}\, T\bar{u}\, dG = 0,$$

i.e. $\bar{u} \equiv v \equiv 0$.

The uniqueness of the solution of problem (11)–(7) follows from Lemma 1.

We now return to the main problem for equation (4).

Lemma 3. If $u \in W^{2,1}_{2,0(y)}(\Omega)$, then

$$\|u\|_{W^{2,1}_{2(y)}(\Omega)} \leqslant c\|Lu\|_{L_2(\Omega)}.$$

The proof is carried out in the same way as in work [2] in establishing an analogous inequality in the case of the heat equation.

Theorem 2. If the domain G belongs to $A_1 \cap A_2$, then the main boundary-value problem (5) for equation (4) is uniquely solvable in $W^{2,1}_{2,0(y)}$ for any right-hand side from $L_2(\Omega)$.

Uniqueness follows from Lemma 3.

Solvability of problem (4)–(5) is proved by means of a density lemma analogous to Lemma 2. In proving this lemma, as $u(x,y,t) \in W^{2}_{2,0(y)}(\Omega)$ one takes the solution of the equation $c_1 u_{xx} + u_{yy} - u_t = v,\ c_1 = \max y,\ (x,y) \in G$, satisfying the boundary condition (5).

We now consider the equation

$$L_1 u \equiv Lu + au_x + bu_y + cu = f(x,y,t), \tag{13}$$

where $a_x, b_y, c \in C^{(0,0)}(\Omega),\ f \in L_2(\Omega)$.

Theorem 3. If the domain G belongs to $A_1 \cap A_2$ and $a = o(^{1/2}),\ -\frac{1}{2}a_x + \frac{1}{2}b_y - c \geqslant \nu > -\frac{1}{4}y_{\max}$ in Ω, then the main boundary-value problem (6) for equation (13) is uniquely solvable in $W^{2,1}_{2,0(y)}(\Omega)$ for any right-hand side f from $L_2(\Omega)$.

The proof of Theorem 3 is built on the basis of Theorem 2 by applying the known method of continuation with respect to a parameter.

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Note: Figure translations are in progress. See original paper for figures.