EXISTENCE OF A SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR AN ORDINARY NONLINEAR DIFFERENTIAL EQUATION OF ORDER n

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Abstract

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EXISTENCE OF A SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR AN ORDINARY NONLINEAR DIFFERENTIAL EQUATION OF ORDER n

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The two-point boundary-value problem for nonlinear ordinary differential equations of the second order has been studied comparatively well, beginning with the classical work of S. N. Bernstein ([1]). The analogous problem for equations of order higher than the second has, in the general case, been little studied (see ([2,3]), where further bibliography is given).

Here we shall consider the boundary-value problem

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}); \tag{1}$$

$$y(a) = \alpha_0, \quad y'(a) = \alpha_1, \dots, \quad y^{(n-2)}(a) = \alpha_{n-2}; \tag{2}$$

$$y^{(k)}(b) = \beta, \tag{3}$$

where the function f is given and continuous for $x \in [a, b]$ $(a < b)$, $-\infty < y, y', \dots, y^{(n-1)} < \infty$; $a, b, \alpha_0, \dots, \alpha_{n-2}, k, \beta$ are given; $0 \le k \le n-1$; all quantities are finite and real. Analogous methods can be applied to boundary conditions of a more general form.

Theorem 1. *Let equation (1) be such that the totality of all its solutions* $y(x)$ $(x \in [a_y, b_y] \subseteq [a, b])$ *has the following properties:*

A. *For every* $M > 0$ *there exists an* $N > 0$ *such that if*

$$|y(a_y)| < M, \quad |y'(a_y)| < M, \dots, \quad |y^{(n-1)}(a_y)| < M, \quad |y^{(k)}(b_y)| < M,$$

then

$$|y^{(k)}(x)| < N \quad (a_y \le x \le b_y).$$

B. *For every $M > 0$ there exists an $N > 0$ such that if*

$$|y(a_y)| < M, \quad |y'(a_y)| < M, \dots, \quad |y^{(n-1)}(a_y)| < M, \quad |y^{(k)}(x)| < M,$$

then $(a_y \le x \le b_y)$,

$$|y^{(n-1)}(x)| < N \quad (a_y \le x \le b_y).$$

C. *For any $a_y < b$, $M_1 > 0$, $M_2 > 0$ there exists an $N > 0$ such that if*

$$|y(a_y)| < M_1, \dots, \quad |y^{(n-2)}(a_y)| < M_1, \quad |y^{(n-1)}(a_y)| > N,$$

then there exists a continuation $\bar{y}(x)$ $(a_y \le x \le b_y^-;\ b_y \le b_y^- \le b)$ of the solution $y(x)$ and a point $c \in [a_y, b_y^-]$ such that

$$|\bar{y}^{(i)}(c)| > M_2, \quad \operatorname{sign} \bar{y}^{(i)}(c) = \operatorname{sign} y^{(n-1)}(a_y)$$

$$(i = k, \dots, n-1).$$

D. *For every M there exists an $N > 0$ such that if*

$$|y^{(k)}(a_y)| > N, \dots, \quad |y^{(n-1)}(a_y)| > N,$$

and all $y^{(k)}(a_y), \dots, y^{(n-1)}(a_y)$ have the same sign, then

$$|y^{(k)}(b_y)| > M.$$

Then problem (1)–(3) has at least one solution.

For the proof of this theorem one must add to (2) the condition

$$y^{(n-1)}(a) = \alpha$$

and consider the continuation of the resulting Cauchy problem as α varies continuously.

Remark. In conditions A–C it is sufficient to restrict oneself to solutions for which $a_y = a$ and the equalities (2) hold.

Conditions A–D are not sufficiently effective, and therefore we shall give a number of sufficient criteria for their fulfillment, expressed directly in terms of the properties of the function f.

Theorem 2. *If for every $M > 0$ there exists a function nondecreasing in all variables $\varphi_M(y_{k+1}, \dots, y_{n-1})$ $(y_{k+1}, \dots, y_{n-1} \ge 0)$, for which*

second,

$$|f(x, y_0, \dots, y_{n-1})| \le \varphi_M \left(|y_{k+1}|, \dots, |y_{n-1}|\right) \tag{4}$$

for $|y_i| < M$ $(i = 0, \dots, k)$, $-\infty < y_{k+1}, \dots, y_{n-1} < \infty$,

$$\lim_{y\to\infty} \varphi_M(y, y^2, \dots, y^{n-k-1}) y^{k-n} = 0,$$

then conditions B and C are satisfied.

The proof is based on estimating $y^{(k+1)}, \dots, y^{(n-1)}$ in terms of $y^{(k)}$ and $y^{(n)}$ by means of the Kolmogorov-Gorny inequality, similarly to the proof of the main theorem of the paper (4).

Remark. For $k = n - 1$, conditions B and C are satisfied automatically. For $k = n - 2$, inequality (4) may be weakened by replacing the right-hand side by $\varphi_M(|y_{n-1}|)$, where $\varphi_M(s)$ $(s_0 < s < \infty)$ is an arbitrary nondecreasing function for which

$$\int^{\infty} s[\varphi_M(s)]^{-1}\, ds = \infty$$

(this is Nagumo' s condition (5)); in particular, one may take $\varphi_M(s) = A^2_{Ms}$ (this is Bernstein' s condition).

Theorem 3. *Suppose equation (1) satisfies condition A for some k; then this condition is also fulfilled for any $k' < k$. The same is true for condition D.*

The proof is easily carried out for $k' = k - 1$, whence the assertion for any k' follows.

We shall say that the equation

$$y^{(p)} = \varphi(x, y, y', \dots, y^{(p-1)}; \eta_1, \dots, \eta_r) \tag{5}$$

with continuous right-hand side containing the parameters $\eta_1, \dots, \eta_r$ $(a \le x \le b; -\infty < y, \dots, y^{(p-1)}; \eta_1, \dots, \eta_r < \infty)$, satisfies condition $\overline{A}$ if for every $M > 0$ there exists an $N > 0$ such that if a function $y(x) \in C_p[a_y, b_y]$ and continuous functions $\eta_1(x), \dots, \eta_r(x)$ $(a_y \le x \le b_y)$ satisfy the relations

$$y^{(p)}(x) = \varphi(x, y(x), \dots, y^{(p-1)}(x); \eta_1(x), \dots, \eta_r(x)) \quad (a_y \le x \le b_y),$$

$$|y(a_y)| < M, \dots, |y^{(p-1)}(a_y)| < M, \quad |y(b_y)| < M,$$

then $|y(x)| < N$ $(a_y \le x \le b_y)$. Analogously, from condition D one obtains condition $\overline{D}$ for equation (5). (In condition $\overline{A}$ it is sufficient to take $a_y = a$.) From Theorem 3 it immediately follows:

Theorem 4. *In order that equation (1) satisfy conditions A and D, it is sufficient that, for some natural number $p \le n - k$, the equation*

$$y^{(p)} = f(x, \eta_1, \dots, \eta_{n-p}, y, y', \dots, y^{(n-p)})$$

satisfy conditions $\overline{A}$ and $\overline{D}$.

Thus, for an equation of high order it becomes possible to apply criteria obtained for equations of lower orders. We indicate some criteria sufficient for satisfying conditions $\overline{A}$ and $\overline{D}$ for equations of the first three orders, denoting, for brevity, $\eta = (\eta_1, \dots, \eta_r)$.

Theorem 5. *Suppose the right-hand side of the equation*

$$y' = \varphi(x, y; \eta) \tag{6}$$

is such that there exist two sequences $u_i(x), v_i(x)$ $(a \le x \le b;\ i = 1, 2, \dots)$ of continuously differentiable functions for which

$$u_i' = \psi_{1i}(x, u_i) < \varphi(x, u_i; \eta), \qquad \min_{[a,b]} u_i(x) \to \infty \quad \text{as } i \to \infty,$$

$$v_i' = \psi_{2i}(x, v_i) > \varphi(x, v_i; \eta), \qquad \max_{[a,b]} v_i(x) \to -\infty \quad \text{as } i \to \infty.$$

Then equation (6) satisfies conditions $\overline{A}$ and $\overline{D}$.

The proof is obtained by comparing y with u_i and v_i. We note that in the formulation of the theorem the functions ψ_i are completely arbitrary; by choosing specific functions ψ_i, one can obtain various classes of functions φ for which conditions $\overline{\mathrm{A}}$ and $\overline{\mathrm{D}}$ are satisfied. Thus, one may set $\psi_{1i} = -c_1 u_i - c_2,\ \psi_{2i} = -c_1 v_i + c_2$ $(c_{1,2} \ge 0)$.

Theorem 6. *Let the right-hand side of the equation*

$$y'' = \varphi(x, y, y'; \eta) \tag{7}$$

be such that there exist two sequences $u_i(x), v_i(x)$ $(a \le x \le b;\ i = 1, 2, \dots)$ of twice continuously differentiable functions for which

$$u_i'' = \psi_{1i}(x, u_i, u_i') < \varphi(x, y, u_i'; \eta) \quad (y \ge u_i), \qquad \min_{[a,b]} u_i(x) \to \infty \quad \text{as } i \to \infty,$$

$$v_i'' = \psi_{2i}(x, v_i, v_i') > \varphi(x, y, v_i'; \eta) \quad (y \le v_i), \qquad \max_{[a,b]} v_i(x) \to -\infty \quad \text{as } i \to \infty.$$

Then equation (7) satisfies conditions $\overline{\mathrm{A}}$ and $\overline{\mathrm{D}}$.

The proof is obtained by comparing y with u_i and v_i. In particular, one may set $\psi_{1i} = c_1 u_i' - c_2,\ \psi_{2i} = c_1 v_i' + c_2$ $(c_{1,2} \ge 0)$.

Theorem 7. *Let the right-hand side of equation (7) be such that there exist two sequences* $u_i(x), v_i(x)$ $(a \le x \le b;\ i = 1, 2, \dots)$ *of continuously differentiable functions for which* $u_i' = \psi_{1i}(x, u_i)$, $v_i' = \psi_{2i}(x, v_i)$; ψ_{1i}, ψ_{2i} *are continuously differentiable in both variables,* $\min_{[a,b]} u_i(x) \to \infty$ *as* $i \to \infty$, $\max_{[a,b]} v_i(x) \to -\infty$ *as* $i \to \infty$, *and*

$$\psi'_{1ix}(x, u_i) + \psi'_{1iu}(x, u_i)\psi_{1i}(x, u_i) < \varphi(x, y, \psi_{1i}(x, u_i); \eta) \quad (y \ge u_i),$$

$$\psi'_{2ix}(x, v_i) + \psi'_{2iv}(x, v_i)\psi_{2i}(x, v_i) > \varphi(x, y, \psi_{2i}(x, v_i); \eta) \quad (y \le v_i). \tag{8}$$

Then equation (7) satisfies conditions $\overline{\mathrm{A}}$ *and* $\overline{\mathrm{D}}$.

The proof is obtained by comparing y with u_i and v_i. Condition (8) is simplified if ψ_i does not depend on x or y.

Theorem 8. *If the right-hand side of the equation*

$$y''' = \varphi(x, y, y', y''; \eta)$$

for all values of the arguments satisfies the inequality $y\varphi(x, y, y_1, y_2; \eta) \ge 0$, *then this equation satisfies conditions* $\overline{\mathrm{A}}$ *and* $\overline{\mathrm{D}}$.

The proof is based on the application of the elementary identity

$$y(x_2) = y(x_1) + \frac{1}{2}\left[y'(x_1) + y'(x_2)\right](x_2 - x_1) + \frac{1}{2}\int_{x_1}^{x_2} (x - x_1)(x - x_2) y'''(x)\, dx$$

Remark. From Theorems 5, 6, and 8, in particular, it follows that if for equation (5) the inequality

$$y\varphi(x, y, y_1, \dots, y_{p-1}; \eta) \ge 0,$$

is satisfied, then for $p = 1, 2, 3$ conditions $\overline{\mathrm{A}}$ and $\overline{\mathrm{D}}$ are satisfied. It is noteworthy that for $p \ge 5$ this is, generally speaking, not so (the case $p = 4$ remains open). Namely, it is verified directly that the function

$$v(x) = (-x)^{-p+1} \sin[\beta \ln((-x))] \qquad (-1 \le x \le 0),$$

where $\beta = \beta_p$ is chosen so that

$$\sum_{i=1}^{p} \arccos \frac{p - 2 + i}{\sqrt{(p - 2 + i)^2 + \beta^2}} = 2\pi,$$

satisfies equations of the form

$$y^{(p)} = \psi_1(x)(y^2 + |y^{(p-1)}|)y \quad \text{or} \quad y^{(p)} = \psi_2(x)(y^2 + |y^{(p-2)}|^{(2p-2)/(2p-3)})y, \quad (9)$$

where $\psi_i \geq 0$, $\psi_i \in C^{p-3}[-1,0]$. Here condition A is not satisfied. The boundary-value problem for the first of equations (9)

$$(y-1) = v(-1), \dots, \; y^{(p-2)}(-1) = v^{(p-2)}(-1), \; y^k(0) = \beta$$

has no solution for any k and β, although the conditions of Theorem 2 are satisfied.

Generalization of the main problem. The following more general problem for equation (1) is studied analogously. Instead of conditions (2), it is required that, for a solution $y(x)$, the line $(y(x), y'(x), \dots, y^{(n-1)}(x))$ in the space $(x, y, y', \dots, y^{(n-1)})$ contain at least one point of a given nonempty connected set Φ (the solution $y(x)$ passes through Φ); instead of condition (3), it is required that this solution be extendable up to $x = b$ and that $F(y(b), y'(b), \dots, y^{(n-1)}(b)) = 0$, where $F(y, y_1, \dots, y_{n-1})$ is a continuous function given for all $y, y_1, \dots, y_{n-1}$. For the existence of at least one solution of this problem it is sufficient that, for some $k = 0, \dots, n-1$, conditions A and B be satisfied and that there exist two solutions y_1, y_2, passing through Φ and extendable up to $x = b$, for which, at $x = b$, $F(y_1, \dots, y_1^{(n-1)}) \geq 0$, $F(y_2, \dots, y_2^{(n-1)}) \leq 0$. To verify conditions A and B one may use Theorems 2–8, and for the fulfillment of the last condition $F \gtrless 0$ it is sufficient, under the hypotheses of Theorem 2, that condition D be fulfilled; for example, that $y_{kF}(y, y_1, \dots, y_{n-1}) > 0$ for all sufficiently large $|y_k|$, and that there exist a double sequence of points $(x_p, y_p, y_{1p}, \dots, y_{n-1,p}) \in \Phi$ $(p = \pm 1, \pm 2, \dots)$, for which

$$\sup_p x_p < b, \quad \sup_p |y_{mp}| < \infty \quad (m = 0, \dots, k),$$

$$\sup_p |y_{mp}|^{\,n-k-1} |y_{n-1,p}|^{\,k-m} < \infty \quad (m = k+1, \dots, n-2),$$

$$\lim_{p \to -\infty} y_{n-1,p} = \infty, \qquad \lim_{p \to -\infty} y_{n-1,p} = -\infty.$$

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Note: Figure translations are in progress. See original paper for figures.