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ON LIMITS OF MOLECULAR MEASURES IN SOME SPACES

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In the works of Katětov ([1–3]), Raikov ([4]), and Ptak ([5]), the following general scheme was studied. On a set X one considers a certain linear space $P(X)$ of scalar functions separating the points of X (the field of scalars is R or C). Let $L(X)$ be the free linear space of the set X over the given field of scalars. The elements of $L(X)$ are formal linear combinations $u = \sum_{x \in X} \lambda_x x$, where the λ_x are scalar coefficients different from zero for only a finite number of points of X. The expression $\langle u, f \rangle = \sum_{x \in X} \lambda_x f(x)$ defines on the product $L(X) \times P(X)$ a bilinear form, by means of which a duality is established between $L(X)$ and $P(X)$. Using this duality, one can topologize $L(X)$ in various ways.

But the elements of the space $L(X)$ may be treated as measures on the set X, concentrated at a finite number of points $x_1, \dots, x_n$ and equal there, respectively, to $\lambda_{x_1}, \dots, \lambda_{x_n}$. Such measures we shall call **molecular**. In the present note we investigate the possibility of approximating a sufficiently broad class of measures by means of molecular measures under a suitably chosen topology, as well as the question of how far all objects obtained in this approximation process may be identified with measures.

We shall say that a system $\mathcal{H} = \{H_\alpha\}_{\alpha \in A}$ of subsets of $P(X)$ is b-closed if: 1) it forms a covering of the space $P(X)$; 2) it is closed with respect to the operations of taking subsets, homotheties of a set, the algebraic sum of two sets, and taking the absolutely convex hull of a set; 3) for every $x \in X$ and every $\alpha \in A$ the set $\{f(x)\}_{f \in H_\alpha}$ is bounded. If the system $\mathcal{H}$ is b-closed, then the system of sets

$$U_{\alpha,\varepsilon} = \{u \in L(X) : |\langle u, f \rangle| < \varepsilon \text{ for all } f \in H_\alpha\}$$

$(\alpha \in A,\ \varepsilon > 0)$ forms a base of neighborhoods of zero in a separated locally convex topology τ_H on $L(X)$.

Let on the set X there be singled out a certain class M of scalar measures, containing all molecular measures, forming a linear space and such that for every measure $\mu \in M$ and every function $f \in P(X)$ there exists the integral $\int_X f\, d\mu$, and moreover the set $\left\{\int_X f\, d\mu\right\}_{f \in H_\alpha}$ is bounded for every $\alpha \in A$. In that case the system of sets

$$\widetilde{U}_{\alpha,\varepsilon} = \{\mu \in M : |\int_X f\,d\mu| < \varepsilon \text{ for all } f \in H_\alpha\}$$

$$(\alpha \in A,\ \varepsilon > 0)$$

forms a base of neighborhoods of zero in a separated locally convex topology τ_H on M, and $(L(X), \tau_H)$ is a topological linear subspace of (M, τ_H). If, moreover, the completion $\widehat{L}$ (or the bounded completion $\widehat{L}$) of the space $(L(X), \tau_H)$ coincides

as a linear space with M, then we shall say that $\hat{L}$ (respectively, $\check{L}$) is canonically isomorphic to M. Under this isomorphism, to an element $\hat{u} \in \hat{L}$ (respectively, $\check{L}$) there corresponds a measure $\mu \in M$ such that

$$\langle \hat{u}, f \rangle = \int_X f\,d\mu$$

for every function $f \in P(X)$.

As usual, for a measure μ defined on some σ-algebra of subsets of X, by $|\mu|(B)$ we shall denote the variation of μ on the set B, and we shall call the measure bounded if $|\mu|(X) < \infty$. A Borel measure μ on a completely regular topological space X will be called regular if for every Borel set B (in particular, for all of X)

$$|\mu|(B) = \sup_{T \subset B} |\mu|(T),$$

where T are bicompact sets. The notation $Rb(X)$ is used for the space of all bounded regular Borel measures on X.

Theorem 1. *Let X be bicompact. Let $P(X)$ be taken to be the space $C(X)$ of all continuous functions on X, and let $\mathcal{H}$ be any b-closed family of $\sigma(c, c')$-bicompact sets containing all sequences uniformly converging to zero. In that case the completion $\hat{L}$ of the space $(L(X), \tau_H)$ is canonically isomorphic to $Rb(X)$.*

Remark. Special cases of this theorem were considered by Katetov ([2],[3]).

We shall say that a set H of continuous functions on a topological space X has the k-property if it is uniformly bounded on X and quasi-equicontinuous ([6]) on every bicompact $T \subset X$. If the original space X is bicompact, this is equivalent to $\sigma(c, c')$-bicompactness of the set H ([6]). If X is a k-space, then this induces bicompactness of the set H in the topology of pointwise convergence on X.

Theorem 2. *Let X be a completely regular space. Let $P(X)$ be taken to be any subspace of the space of all bounded continuous functions on X such that, for every bicompact $T \subset X$, the restrictions of all functions $f \in P(X)$ to T form the entire space $C(T)$. Let $\mathcal{H}$ be a b-closed system of subsets of $P(X)$ having the k-property. Then $Rb(X) \subset \hat{L}$, where $\hat{L}$ is the completion of the space $(L(X), \tau_H)$.*

Consequence. Let (X, ξ) be a separated uniform space, $P(X)$ the space of all bounded uniformly continuous functions on X. Let $\mathcal{H}$ be a b-closed system of subsets of $P(X)$ having the k-property. Then $Rb(X) \subset \hat{L}$.

We shall say that a measure $\mu \in Rb(X)$ is τ_H-approximable if there exists a sequence $u_1, \dots, u_n, \dots$ of elements of $L(X)$, τ_H-converging to μ.

Theorem 3. *Let X be a completely regular space such that every bicompact $T \subset X$ is metrizable. Let $P(X)$ satisfy the conditions of Theorem 2; let $\mathcal{H}$ be a b-closed system, each set in which is uniformly bounded on X and equicontinuous at every point $x \in X$. Then every measure $\mu \in Rb(X)$ is τ_H-approximable.*

Theorem 4. *Let X be taken to be the finite-dimensional Euclidean space E^n. Let $P_i(E^n)$ be the space of all bounded uniformly continuous functions on E^n. Let $\mathcal{H}$ be a b-closed system of subsets of $P(E^n)$ such that every set $H \in \mathcal{H}$ is uniformly bounded and uniformly equicontinuous on E^n, and moreover all countable sets of the indicated kind are contained in $\mathcal{H}$. In that case the completion $\hat{L}(E^n)$ of the space $(L(E^n), \tau_H)$ is canonically isomorphic to the space $Rb(E^n)$.*

If (X, ξ) is a separated uniform space; $P(X)$ is the space of all uniformly continuous functions on (X, ξ); $\mathcal{H}$ is a family

of all subsets of $P(X)$, pointwise bounded on X and equicontinuously uniformly continuous, then $(L(X), \tau_H)$ is a free locally convex space of the original space (X, ξ) ([4]).

A measure $\mu \in Rb(E^n)$ is said to have a first moment if

$$\int_X |x| \, d\mu < \infty,$$

where $|x|$ is the Euclidean norm of the element $x \in E^n$. Denote by $\widetilde{Rb}(X)$ the space of all such measures.

Theorem 5. *The completion of the free locally convex space of a finite-dimensional Euclidean space E^n is canonically isomorphic to the space $\widetilde{Rb}(X)$.*

Recall that the **bounded completion** of a topological linear space (E, τ) is the smallest boundedly complete (in other terminology, quasi-complete ([7])) space $(\breve{E}, \breve{\tau})$ containing (E, τ) as an everywhere dense topological and linear subspace.

A uniform space (X, ξ) is called **uniformly locally bicompact** if its uniformity ξ has a base v of entourages such that, for every $V \in v$ and for every $x \in X$, the set $V[x]$ is bicompact.

Theorem 6. *Let (X, ξ) be a separable uniformly locally bicompact space; $P(X)$ the space of all bounded uniformly continuous functions on (X, ξ); $\mathcal{H}$ the family of all uniformly bounded equicontinuously uniformly continuous sets from $P(X)$. In this case the bounded completion $\breve{L}$ of the space $(L(X), \tau_H)$ is canonically isomorphic to the space $Rb(X)$.*

Theorem $6'$**.** *Let X be a paracompact locally bicompact topological space; $P(X)$ the space of all bounded continuous functions on X; $\mathcal{H}$ the family of all subsets of $P(X)$ uniformly bounded on X and equicontinuous at every point $x \in X$. In this case the bounded completion $\breve{L}$ of the space $(L(X), \tau_H)$ is canonically isomorphic to the space $Rb(X)$.*

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Note: Figure translations are in progress. See original paper for figures.