ON THE COMPLETENESS OF A SYSTEM OF ANALYTIC FUNCTIONS \(\{z^n F^{(n)}(\lambda_n z)\}\)

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Abstract

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I. I. IBRAGIMOV, Ch. G. AGAMALIEV

ON THE COMPLETENESS OF A SYSTEM OF ANALYTIC FUNCTIONS $\{z^n F^{(n)}(\lambda_n z)\}$

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A. I. Markushevich ([2]), by a method of functional analysis, considered the completeness of the system of functions* $\{z^n F^{(n)}(\lambda_n z)\}$ in the case when the sequence of complex numbers $\{\lambda_n\}$, converging to ξ_0, $|\xi_0| < 1$, is such that

$$\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < +\infty,$$

and $F(z)$ is an analytic function in the finite disk $|z| < R$, or an entire analytic function of exponential type σ. In particular, the completeness of the system of functions $\{z^n e^{\lambda_n z}\}$ was studied in ([3]) also under the assumption that $|\lambda_n| \leq 1$ $(n = 0, 1, 2, \ldots)$.

In the present article the completeness of the system of functions $\{z^n F^{(n)}(\lambda_n z)\}$ is studied by means of the Abel-Goncharov interpolation formula in the case when $F(z)$ is an arbitrary entire analytic function, and the sequence of complex numbers $\{\lambda_n\}$ is such that

$$|\lambda_0| \leq |\lambda_1| \leq \ldots \leq |\lambda_n| \leq \ldots; \qquad \lim_{n\to\infty} |\lambda_n| = \infty. \tag{1}$$

Moreover, by the same interpolation method a number of earlier known results are obtained. This shows that the methods of interpolation theory are effective methods in investigations of the completeness of systems of analytic functions, as was first noted by A. O. Gelfond ([1]).

Let us note that if $F(z)$ is an entire function, then $F(z\xi)$ is also an entire function in each of the variables z and ξ. Further, let us note that the Abel-Goncharov interpolation formula for the function $F(z\xi)$ with respect to the variable ξ, with nodes $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ (λ_n satisfy condition (1)), has the form:

$$F(z\xi) = F(\lambda_0 z) + zF'(\lambda_1 z)P_1(\xi) +$$

$$+z^2F''(\lambda_2 z)P_2(\xi) + ... + z^{nF^{(n)}}(\lambda_n z)P_n(\xi) + R_n(z,\xi), \tag{2}$$

where $P_0(\xi) \equiv 1$,

$$P_k(\xi) = \int_{\lambda_0}^{\xi}\int_{\lambda_1}^{\xi_1} ... \int_{\lambda_{k-1}}^{\xi_{k-1}} d\xi_1\, d\xi_2 ... d\xi_k \quad (k = 1, 2, ...); \tag{3}$$

$$R_n(z,\xi) = z^{n+1}\int_{\lambda_0}^{\xi}\int_{\lambda_1}^{\xi_1} ... \int_{\lambda_n}^{\xi_n} F^{(n+1)}(\xi_{n+1}z)\, d\xi_1 ... d\xi_{n+1}. \tag{4}$$

Differentiating equality (2) m times with respect to ξ and then putting $\xi = 0$, we obtain

$$z^{mF^{(m)}}(0) = \sum_{k=m}^{n} P_k^{(m)}(0)\, z^k F^{(k)}(\lambda_k z) + R_{n,m}(z), \tag{5}$$

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* The system of functions $\{z^{nF^{(n)}}(\lambda_n z)\}$ is a generalization of the system of functions $\{z^{nF^{(n)}}(z)\}$, whose completeness was studied in ([3]).

where

$$P_k^{(m)}(0) = \int_{\lambda_{m-1}}^{0}\int_{\lambda_m}^{\xi_m} \cdots \int_{\lambda_{k-1}}^{\xi_{k-1}} d\xi_m \cdots d\xi_k \quad (m = 0, 1, 2, ... ;\ k = 0, 1, 2, ...); \tag{6}$$

$$R_{n,m}(z) = z^{n+1}\int_{\lambda_{m-1}}^{0}\int_{\lambda_m}^{\xi_m} \cdots \int_{\lambda_n}^{\xi_n} F^{(n+1)}(\xi_{n+1}z)\, d\xi_m \cdots d\xi_{n+1}. \tag{7}$$

Equality (5) shows that the system of analytic functions $\{z^{nF^{(n)}}(\lambda_n z)\}$ is complete under those conditions for which $|R_{n,m}(z)|$ tends to zero uniformly in z as $n \to \infty$ and, moreover, all Taylor coefficients of the function $F(z)$ must be nonzero: $F^{(m)}(0) \neq 0$ $(m = 0, 1, 2, ...)$.

Let $F(z)$ be an entire function with maximum modulus $M(r) = \max_{|z|\leq r} |F(z)|$, and let the numbers λ_n $(n = 0, 1, 2, ...)$ satisfy conditions (1). Denote by $n(r)$ the number of points of the sequence

$$s_n = \sum_{k=0}^{n} |\lambda_{k+1} - \lambda_k| \quad (n = 1, 2, 3, ...),$$

lying inside the segment $[0, r]$.

We note that in this case $|R_{n,m}(z)|$ tends to zero uniformly in z $(|z| \le \rho)$ as $n \to \infty$ whenever $|R_n(z, \xi)|$, defined by equality (4), tends to zero uniformly in z $(|z| \le \rho)$ and in ξ $(|\xi| \le R)$ as $n \to \infty$. The latter is possible, as shown in paper ([4]), if for $M(r)$ and $n(r)$ the inequality

$$\ln M(F; \rho R/\theta) < C(\theta) n(\rho R), \tag{8}$$

holds, where

$$C(\theta) < \ln \frac{1-\theta}{\theta}, \qquad 0 < \theta < 1/2. \tag{9}$$

Theorem 1. If $F(z)$ is an entire function with maximum modulus $M(F; r)$ in the disk $|z| \le r$, with nonzero Taylor coefficients $(F^{(m)}(0) \ne 0,\ m = 0, 1, 2, ...)$, and $n(r)$ is the number of points of the sequence $\{s_n\}$ inside the segment $[0, r]$, then the system of functions $\{z^{nF^{(n)}}(\lambda_n z)\}$ is complete in every finite disk $|z| \le \rho$, if inequalities (8) and (9) are satisfied.

Corollary 1. If μ is the lower density of the sequence $\{s_n\}$, i.e.

$$\mu = \varliminf_{r \to \infty} \frac{n(r)}{r}, \tag{10}$$

and ω is the positive root of the equation*

$$\omega e^{\omega + 1} = 1, \tag{11}$$

then the system $\{z^n e^{\lambda_n z}\}$ is complete in every finite disk $|z| \le \rho$ under the condition $\mu > 1/\omega$.

Corollary 2. If $F(z)$ is an entire function of order ρ_1, type σ, with nonzero Taylor coefficients $(F^{(m)}(0) \ne 0,\ m = 0, 1, 2 ...)$, and the numbers s_n satisfy the condition

$$\varliminf_{n \to \infty} \frac{s_n}{n^{1/\rho}} = \tau \quad (\rho > 0,\ \tau > 0), \tag{12}$$

* The number ω was introduced for consideration by V. L. Goncharov in article ([5a]); one should also note the translation of the content of this article included in book ([5b]).

then the system of functions $\{z^n F^{(n)}(\lambda_n z)\}$ is complete in every finite disk $|z| \le \rho$ under the conditions: 1) if $\rho < \rho_1$, or 2) if $\rho = \rho_1$ and

$$\sigma < \left(\frac{\omega}{\tau}\right)^{\rho} \frac{(\omega + 1)^{1-\rho}}{\rho}, \tag{13}$$

where ω is the positive root of equation (11).

Corollary 3. If the entire function $F(z)$, with Taylor coefficients different from zero, satisfies the condition

$$\lim_{r\to\infty} \frac{\ln\ln M(F,r)}{r^\rho} = A \qquad (A \geq 0)$$

and the numbers s_n satisfy the condition

$$\lim_{n\to\infty} \frac{s_n}{(\ln n)^{1/\rho}} = \tau \qquad (\rho > 0,\ \tau > 0),$$

then the system of functions $\{z^n F^{(n)}(\lambda_n z)\}$ is complete in every finite disk whenever the inequality

$$A\tau^\rho < (1/2)^\rho$$

is satisfied.

We note that the same interpolation method can be used to prove a number of previously known results (see ([2])).

Theorem 2. If $F(z)$ is an entire function of exponential type σ with Taylor coefficients different from zero, then the system of functions $\{z^n F^{(n)}(\lambda_n z)\}$, for $|\lambda_n| \leq 1$ $(n = 0, 1, 2, ...)$, is complete in the disk $|z| < \ln 2/\sigma$.

This theorem is a generalization of the fact that the system of functions $\{z^n e^{\lambda_n z}\}$, for $|\lambda_n| \leq 1$, is complete in the disk $|z| < \ln 2$ (see ([3])).

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References

[1] A. O. Gelfond, *Matem. sborn.*, 4 (46), 1, 149 (1939).
[2] A. I. Markushevich, *Matem. sborn.*, 17 (59), 2, 211 (1945).
[3] I. I. Ibragimov, *Izv. AN SSSR, ser. matem.*, 3, 553 (1939).
[4] I. I. Ibragimov, *Matem. sborn.*, 21 (63), 1, 49 (1947).
[5] V. L. Goncharov, a) *Ann. Éc. Norm. sup.*, 74, 4 (1930); b) *Theory of Interpolation and Approximation of Functions*, 1st ed., Moscow—Leningrad, 1934.

Note: Figure translations are in progress. See original paper for figures.