

# FOURIER SERIES IN EIGENFUNCTIONS OF MULTIDIMENSIONAL DOMAINS, DIVERGING ALMOST EVERYWHERE

MATHEMATICS

1966

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**Abstract**

**Full Text**

UDC 517.521.5

*MATHEMATICS*

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## **FOURIER SERIES IN EIGENFUNCTIONS OF MULTIDIMENSIONAL DOMAINS, DIVERGING ALMOST EVERYWHERE**

*(Presented by Academician A. N. Kolmogorov, 22 XII 1965)*

In 1923 A. N. Kolmogorov <sup>(1)</sup> constructed an example of a function  $f(x)$  belonging to the class  $L_1[0 \leq x \leq 2\pi]$  and having a trigonometric Fourier series that diverges almost everywhere on the segment  $0 \leq x \leq 2\pi$ .\* To this day there remains open the question of the existence of a function  $f(x)$  from the class  $L_p[0 \leq x \leq 2\pi]$ , with  $p > 1$ , having a trigonometric Fourier series that diverges on a set of positive measure. The question has not at all been studied of what  $L_p$  a function of several variables  $f$ , defined in an  $N$ -dimensional domain and having an almost everywhere divergent Fourier series in that domain with respect to the eigenfunctions of the Laplace operator, may belong to. The present paper is devoted to this question.

We consider the eigenfunctions of the  $N$ -dimensional ball  $\Omega$  of radius  $R$  with center at the point  $x_0$ , satisfying a homogeneous boundary condition of the first or second kind. By the symbol  $K$  we denote a circular  $N$ -dimensional cone having its vertex at the point  $x_0$  and arbitrary fixed height and solid angle at the vertex. We have constructed the following two examples.

**Example 1.** *For any  $N \geq 2$  there exists a function  $f$ , belonging in the ball  $\Omega$  to the class  $L_{2N/(N+1)}$  and such that the Fourier series of this function in the eigenfunctions of the ball  $\Omega$ , under any order of succession of its terms, diverges almost everywhere in the ball  $\Omega$  (the  $n$ -th term of this series does not tend to zero as  $n \rightarrow \infty$  almost everywhere in the ball  $\Omega$ ).*

Let us note that the exponent  $2N/(N+1)$ , for large  $N$ , is arbitrarily close to two. Thus we establish that for a large number  $N$  of dimensions there exists a function  $f$  "almost in  $L_2$ ," whose Fourier series diverges almost everywhere in the domain under consideration.

**Example 2.** *For any  $N \geq 2$  there exists a function  $f$  satisfying the following three requirements: 1)  $f$  belongs to the class  $L_{2N/(N+1)}$  in the ball  $\Omega$ ; 2)  $f \equiv 0$  in a given fixed domain  $D$ , belonging to the difference of the ball  $\Omega$  and the cone  $K$ ; 3) the Fourier series of the function  $f$ , under any order of succession of its*

terms, diverges almost everywhere in the ball  $\Omega$  (including almost everywhere in the domain  $D$ ).

Example 2 shows that if the condition  $f \in L_p$ , where  $p > 2N/(N + 1)$ , is not satisfied everywhere in the domain under consideration, then one cannot expect convergence of the Fourier series of the function  $f$  almost everywhere in a neighborhood of a given point, however smooth the function  $f$  may be in a neighborhood of this point.

We outline the scheme of construction of Examples 1 and 2.

1. The normalized eigenfunctions of the  $N$ -dimensional ball possessing spherical symmetry have the form:

a) for the case of the first boundary-value problem  $u(R) = 0$

$$u_n(r) = \frac{1}{\sqrt{\omega_N}} \frac{\sqrt{2}}{R} \frac{1}{J_{(N-4)/2}(\mu_n)} \frac{J_{(N-2)/2}\left(\frac{r}{R}\mu_n\right)}{r^{(N-2)/2}} \quad (n = 1, 2, \dots), \quad (1)$$

where  $\mu_n$  are the zeros of the Bessel function  $J_{(N-2)/2}(x)$ ;

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\* For a detailed construction of Kolmogorov's example, see <sup>(2)</sup>, pp. 391-402.

b) for the case of the second boundary-value problem  $\frac{du}{dr}(R) = 0$

$$u_n(r) = \frac{1}{\sqrt{\omega_N}} \frac{\sqrt{2}}{R} \frac{1}{J_{(N-2)/2}(\mu_n)} \frac{J_{(N-2)/2}\left(\frac{r}{R}\mu_n\right)}{r^{(N-2)/2}} \quad (n = 1, 2, \dots), \quad (2)$$

where  $\mu_n$  are the zeros of the Bessel function  $J_{N/2}(x)$ ,

$$\omega_N = \frac{2(\sqrt{\pi})^N}{\Gamma(N/2)}.$$

We shall rely on the following lemma of resonance type.

**Lemma 1.** If  $q > 1$ ;  $\Omega$  is an arbitrary domain of any number of dimensions;  $\{u_n(x)\}$  is a sequence of functions each of which belongs to the class  $L_q(\Omega)$ , then from the unboundedness of the numerical sequence

$$\int_{\Omega} |u_n(x)|^q dx \quad (3)$$

there follows the existence of a function  $f(x) \in L_p(\Omega)$ , where  $1/p + 1/q = 1$ , for which the numerical sequence

$$\int_{\Omega} f(x)u_n(x) dx \tag{4}$$

is also unbounded.

A proof of Lemma 1 may be found, for example, in <sup>3</sup> (vol. I, p. 267, theorem 9, 11.III). To construct example 1 we shall apply Lemma 1 to the sequence of eigenfunctions (1) or (2), taking as the domain  $\Omega$  the  $N$ -dimensional ball of radius  $R$  and putting  $q = 2N/(N - 1)$ , where  $N \geq 2$ .

The essential point of the proof is the establishment of the fact that the sequence (3) is unbounded for  $q = 2N/(N - 1)$ . First of all note that from the asymptotics of the Bessel functions and from the definition of the zeros  $\mu_n$  it follows that, for large indices  $n$ , the inequalities hold:

a) for the eigenfunctions of the first boundary-value problem,

$$\frac{1}{2} \leq \sqrt{\frac{2}{\pi} \frac{1}{\sqrt{\mu_n}}} |J_{(N-4)/2}(\mu_n)| \leq \frac{3}{2}. \tag{5}$$

b) for the eigenfunctions of the second boundary-value problem,

$$\frac{1}{2} \leq \sqrt{\frac{2}{\pi} \frac{1}{\sqrt{\mu_n}}} |J_{(N-2)/2}(\mu_n)| \leq \frac{3}{2}. \tag{6}$$

Passing in the integral (3) to spherical coordinates, we may write

$$\int_{\Omega} |u_n(x)|^q dx \geq \omega_N \int_{1/\mu_n}^R |u_n(r)|^q r^{N-1} dr. \tag{7}$$

With the aid of the estimates (5) and (6), the asymptotic formula for the Bessel function

$$J_{(N-2)/2} \left( \frac{r}{R} \mu_n \right) = \sqrt{\frac{2R}{\pi r \mu_n}} \cos \left( \frac{r}{R} \mu_n - \frac{\pi}{2} \frac{N-1}{2} \right) + O \left( \frac{1}{\mu_n^{3/2} r^{3/2}} \right)$$

and the triangle inequality written in the  $L_q$  norm, it is not hard to verify that, in order to establish the unboundedness of the sequence, stand-

appearing on the right-hand side of (7), it is sufficient to prove, first, the unboundedness of the sequence

$$\int_{1/\mu_n}^R \frac{1}{r} \left| \cos \left( \frac{r}{R} \mu_n - \frac{\pi}{2} \frac{N-1}{2} \right) \right|^q dr$$

and, second, the boundedness of the sequence

$$\frac{1}{\mu_n^q} \int_{1/\mu_n}^R \frac{dr}{r^{2N/(N-1)+1}}.$$

Thus the unboundedness of the sequence (3) for  $q = 2N/(N - 1)$  has been established. By Lemma 1 there exists a function  $f(x) \in L_{2N/(N+1)}(\Omega)$  whose sequence of Fourier coefficients (4) is unbounded.

2. We now prove that the Fourier series constructed from the function  $f(x)$  diverges almost everywhere in the ball  $\Omega$ , under any ordering of its terms. It is sufficient to prove that the  $n$ -th term of the Fourier series does not tend to zero as  $n \rightarrow \infty$  almost everywhere in the ball  $\Omega$ .

We shall rely on the following auxiliary assertion.

**Lemma 2.** *For every  $\varepsilon > 0$  there exist a number  $\delta > 0$  and an index  $n_0$  such that, for all  $n \geq n_0$ , each eigenfunction  $u_n(x)$ , everywhere in the ball  $\Omega$ , except for some set of points  $E_n$  of measure not exceeding  $\varepsilon$ , satisfies the inequality*

$$|u_n(x)| \geq \delta.$$

The proof of Lemma 2 is carried out with the aid of the asymptotic formula for the Bessel function and its zeros.

To prove that the  $n$ -th term of the Fourier series constructed from our function  $f(x)$  does not tend to zero as  $n \rightarrow \infty$  almost everywhere in the ball  $\Omega$ , it is enough to prove that *any subsequence of the sequence of eigenfunctions (1) or (2) can be infinitely small in the ball  $\Omega$  only on a set of measure zero.*

Suppose this is not so; that is, suppose that some subsequence  $\{u_{k_n}(x)\}$  is infinitely small on some set  $E$  of positive measure  $4\varepsilon > 0$ . By D. F. Egorov's theorem, there exists a set  $\hat{E}$  of measure not less than  $2\varepsilon$  on which the indicated subsequence  $\{u_{k_n}(x)\}$  converges to zero uniformly, i.e.

$$\lim_{n \rightarrow \infty} \int_{\hat{E}} |u_{k_n}(x)| dx = 0. \quad (8)$$

By Lemma 2, there exist  $\delta > 0$  and an index  $n_0$  such that for  $n \geq n_0$  each eigenfunction  $u_{k_n}(x)$ , everywhere in the ball  $\Omega$ , except for some set of points  $E_n$  of measure not exceeding  $\varepsilon$ , satisfies the inequality

$$|u_{k_n}(x)| \geq \delta. \quad (9)$$

Let  $\hat{E}_n = \hat{E} \cap E_n$ . Then  $\mu(\hat{E}_n) < \varepsilon$ , and hence

$$\mu(\hat{E} - \hat{E}_n) > \varepsilon. \quad (10)$$

Finally, by (9) and (10), for  $k_n \geq n_0$  we obtain

$$\int_{\hat{E}} |u_{k_n}(x)| dx \geq \int_{\hat{E} - \hat{E}_n} |u_{k_n}(x)| dx \geq \varepsilon \delta,$$

which contradicts the existence of the limit (8).

Thus the construction of Example 1 is complete.

3. By an analogous method, the construction of example 2 is carried out. In this case Lemma 1 of Section 1 is applied to the sequence of eigenfunctions (1) or (2) not on the whole ball  $\Omega$ , but only on the cone  $K$ . In this way a function  $f(x)$  is constructed, belonging to the class  $L_{2N/(N+1)}$  in the cone  $K$ . We extend this function to the whole ball  $\Omega$  by setting it equal to zero outside the cone  $K$ . The arguments of Section 2 remain fully valid.

The author expresses gratitude to A. N. Tikhonov and P. L. Ul'yanov for discussing the results of this work.

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Received  
8 XII 1965

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2. N. K. Bari, *Trigonometric Series*, Moscow, 1961.
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*Note: Figure translations are in progress. See original paper for figures.*

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