ON THE EXPRESSION OF A FUNCTION THROUGH ITS INTEGRALS OVER ELLIPSOIDS

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Abstract

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MATHEMATICS

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ON THE EXPRESSION OF A FUNCTION THROUGH ITS INTEGRALS OVER ELLIPSOIDS

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Let

$$P = P(x) = \sum_{s,t=1}^{n} a_{st} x_{sx} t$$

be a positive definite quadratic form in Euclidean space of odd dimension n, and let $\delta^{(p)}(c^2 - P)$ be a functional on the space K of finite infinitely differentiable functions. To a function $f(x)$ from K we associate its integrals over ellipsoids

$$\check{f}_p(x) = \int f(x+y)\delta^{(p)}(c^2 - P(y))\, dy$$

for fixed p, where $dy = dy_1 \dots dy_n$; the integration is carried out over all values of the variables (which is also understood for the other integrals). It is required, knowing $\check{f}_p(x)$, to reconstruct the function $f(x)$.

The following results are obtained in the paper:

1. If $f(x)$ is concentrated inside the domain $P(x) \leq c^2$, then

$$f(x) = (-1)^{(p-q)/2} \frac{2c^2\Delta}{\pi^{n-1}} \int \check{f}_p(x+y)\delta^{(q)}(c^2 - P(y))\, dy, \tag{1}$$

where $\Delta = \det\|a_{st}\|$; q is determined from the condition $p + q = n - 1$ (this equality is assumed to hold also below); $p = 0, 1, \dots, n-1$, and also

$$f(x) = -(-1)^{(p-q)/2} \frac{\Delta}{2\pi^{n-1}} \int L_p \check{f}_{p-1}(x+y)\delta^{(q-1)}(c^2 - P(y))\, dy, \tag{2}$$

where

$$L_p = \sum_{s,t=1}^{n} a^{st} \frac{\partial^2}{\partial x_s \partial x_t};$$

a^{st} are the coefficients of the matrix inverse to $\|a_{st}\|$; $p = 1, 2, \dots, n-2$.

2. If $\check{f}_p(x)$ and $\check{f}_{p-1}(x)$ are known, then

$$f(x) = \frac{(-1)^{(p-q)/2}\Delta}{4\pi^{n-1}} \left[4c^2 \int \check{f}_p(x+y)\delta^{(q)}(c^2 - P(y))\,dy - \int L_p \check{f}_{p-1}(x+y)\delta^{(q-1)}(c^2 - P(y))\,dy\right],$$

where $f(x)$ is an arbitrary function from K, $p = 1, 2, \dots, n-2$.

This establishes the equivalence of (1) and (2) for those values of p for which (2) is defined.

3. For an arbitrary function from K,

$$f(x) = \frac{2\Delta c^2}{\pi^{n-1}} \sum_{k=0}^{\infty} (-1)^k (2k+1) \int \check{f}_{(n-1)/2}(x+y)\delta^{((n-1)/2)}(c^2(2k+1)^2 - P(y))\,dy,$$

$$f(x) = -\frac{\Delta}{2\pi^{n-1}} \sum_{k=0}^{\infty} \int L_P \check{f}_{(n-3)/2}(x+y)\delta^{((n-3)/2)}(c^2(2k+1)^2 - P(y))\,dy, \quad (3)$$

where the sums terminate for sufficiently large k.

Relation (1) for $p = 0$ in the case when P is a sum of squares was obtained earlier by another method by N. Ya. Vilenkin. For $n = 3$, (3) is John' s formula for expressing a function through its spherical means over spheres of fixed radius in an affinely invariant notation ((1), p. 109).

We next prove some relations for delta-functions, from which these results follow directly.

Denote by

$$Q = Q(\sigma) = \sum_{s,t=1}^{n} a^{st} \sigma_s \sigma_t$$

the quadratic form conjugate to P. As is known from (2), the following formulas hold:

$$F\left[\delta^{(p-1)}(c^2-P)\right] = \frac{\pi^{n/2}}{\sqrt{\Delta}}\left(\frac{2c}{Q^{1/2}}\right)^{n/2-p} J_{n/2-p}(cQ^{1/2}), \tag{4}$$

$$F[L_P\delta(x)] = -Q, \tag{5}$$

where F denotes the Fourier transform; $p \geq 1$ is an integer; $c > 0$, $\delta(x) = \delta(x_1, \dots, x_n)$ is the delta-function; $J_{n/2-p}$ is a Bessel function. The left-hand sides of (4) and (5) are understood as functionals on the space Z, dual to K with respect to the Fourier transform.

Consider the generalized function $\delta^{(p)}(c^2-P) * \delta^{(q)}(c^2-P)$, defined by means of convolution. Since the inverse Fourier transform F^{-1} of the function 1 is $\delta(x)$, using the properties of the Fourier transform of a convolution (see ([3])) and (4), and introducing the notation $z = cQ^{1/2}$, $m = (p-q)/2$ (m an integer), we shall have

$$\delta(x) = (-1)^m \frac{2c^2\Delta}{\pi^{n-1}} \delta^{(p)}(c^2-P) * \delta^{(q)}(c^2-P) + A_m(c,P), \tag{6}$$

where

$$A_m(c,P) = F^{-1}\left[1 - (-1)^m \pi z J_{-m-1/2}(z) J_{m-1/2}(z)\right], \qquad p = 0, 1, \dots, n-1.$$

Similarly, taking into account (5), the differentiation rule for convolution, and the equality

$$J_\nu(z)J_{-\nu+1}(z) + J_{-\nu}(z)J_{\nu-1}(z) = \frac{2\sin\nu\pi}{\pi z},$$

one can obtain

$$\delta(x) = -(-1)^m \frac{\Delta}{2\pi^{n-1}} \delta^{(p-1)}(c^2-P) * L_P\delta^{(q-1)}(c^2-P) - A_m(c,P).$$

We shall show that the functional $A_m(c,P)$ is concentrated outside the domain $P(x) < (2c)^2$.

For the proof, using the representation of Bessel functions with half-integral index in terms of Lommel polynomials $R_{m,\nu}(z)$ (([4]), p. 326), we transform the expression for $A_m(c,P)$:

$$F[A_m] = U_m(z)\cos 2z + V_m(z)\sin 2z,$$

where

$$U_m(z) = R_{m-1,\,3/2}(z)R_{m-1,\,1/2}(z) + R_{m,\,1/2}(z)R_{m-2,\,3/2}(z),$$

$$V_m(z) = R_{m-1,\,3/2}(z)R_{m-2,\,3/2}(z) - R_{m,\,1/2}(z)R_{m-1,\,1/2}(z).$$

Here we have used Krell’ s formula for Lommel polynomials ((^{4}), p. 328). The functions $U_m(z)$ and $V_m(z)$ are polynomials in negative powers of z, and the validity of the assertion follows from the following lemma.

Lemma. The functional $F^{-1}[\cos z/z^{2k}]$ for integer $k < n/2$ and the functional $F^{-1}[\sin z/z^{2k+1}]$ for integer $k < (n-1)/2$ are concentrated outside the domain $P(x) < c^2$.

For the proof of the lemma it suffices, using formulas 8.411,7; 6.699,1 and 2 from (^{5}), to establish two relations with a hypergeometric function. We give one of them:

$$z^{\lambda-n}\cos z e^{-i(x,\sigma)}\,d\sigma = \frac{2\sqrt{\Delta}\,\pi^{n/2}}{c^n\Gamma(n/2)}\Gamma(\lambda)\cos\frac{\pi\lambda}{2}F\left(\frac{\lambda}{2},\ \frac{\lambda+1}{2},\ \frac{n}{2};\ \frac{P}{c^2}\right).$$

Here λ is a complex parameter, $d\sigma = d\sigma_1 \ldots d\sigma_n$, $P(x) < c^2$. For $0 < \operatorname{Re}\lambda < (n+1)/2$ the integral converges directly; for other values of $\operatorname{Re}\lambda$ it is understood in the regularized sense, by virtue of the uniqueness of analytic continuation.

Let k be an integer. For the case $m = 0$, analogously to (6), we have

$$F\left[2\Delta c^2\delta^{((n-1)/2)}(c^2-P) * \delta^{((n-1)/2)}(c^2(2k+1)^2-P)\right] =$$

$$= \pi^{n-1}(\cos 2kz + \cos 2(k+1)z),$$

and further

$$\delta(x) = \frac{2\Delta c^2}{\pi^{n-1}}\delta^{((n-1)/2)}(c^2-P)*$$

$$*\sum_{k=0}^{r-1}(-1)^k(2k+1)\delta^{((n-1)/2)}(c^2(2k+1)^2-P) + (-1)^r B_n(cr,P),$$

where $r \geq 1$ is an arbitrary integer,

$$B_n(c,P) = \frac{2c\sqrt{\Delta}}{\pi^{(n-1)/2}}\delta^{((n-1)/2)}(4c^2-P).$$

In the same way one easily obtains

$$\delta(x) = -\frac{\Delta}{2\pi^{n-1}}\delta^{((n-3)/2)}(c^2 - P) * L_P \sum_{k=0}^{r-1} \delta^{((n-3)/2)}(c^2(2k+1)^2 - P) + B_n(cr, P).$$

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Note: Figure translations are in progress. See original paper for figures.

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