ANISOTROPIC BESSEL POTENTIALS.

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Abstract

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ANISOTROPIC BESSEL POTENTIALS.

EMBEDDING THEOREMS FOR THE SOBOLEV SPACE $L_p^{(r_1,\dots,r_n)}$ WITH FRACTIONAL DERIVATIVES

(Presented by Academician S. L. Sobolev on 3 January 1966)

1. Let E_n be the n-dimensional space of points $x = (x_1, \dots, x_n)$.

Definition 1. An S'-distribution f belongs to the space $L_p^{(r_1,\dots,r_n)}(E_n) = L_p^{(r)}$ if its Bessel derivatives of orders $r_1, \dots, r_n$ with respect to the variables $x_1, \dots, x_n$, respectively, belong to $L_p(E_n)$ $(r_j \leq 0,\ j = 1, \dots, n;\ 1 < p < \infty)$.

Here S' is the space of distributions of L. Schwartz (^{13}). By the Bessel derivative $J_{x_j}^{r_j} f$ of the function f of order r_j with respect to the variable x_j is meant the expression

$$J_{x_j}^{r_j} f = F^{-1}\big[(1 + \lambda_j^2)^{r_j/2} F f\big],$$

where F is the Fourier transform operator; $\lambda = (\lambda_1, \dots, \lambda_n)$ is the independent variable in the space of Fourier images. We shall also write $Ff = \tilde{f}(\lambda)$.

Put

$$\|f\|_{L_p^{(r)}} = \sum_{j=1}^{n} \|J_{x_j}^{r_j} f\|_{L_p}.$$

According to Definition 1, from $f \in L_p^{(r)}$ it follows that

$$\sum_{j=1}^{n} (1 + \lambda_j^2)^{r_j/2}\, \tilde{f} = \tilde{g}, \qquad g \in L_p, \quad 1 < p < \infty. \tag{1}$$

Recall that a function $\Phi(\lambda)$ is called a multiplier of type (p, q) (in symbols $\Phi \in M_p^q$) if the transformation $T_\Phi f = F^{-1}[\Phi(\lambda) F f]$ is bounded, that is $\|T_\Phi f\|_{L_q} \leq c\|f\|_{L_p}$. Rewrite equality (1) in the form

$$\tilde f = \frac{1}{\left[\sum_1^n (1+\lambda_j^2)^{a_j}\right]^{\rho/2}} \frac{\left[\sum_1^n (1+\lambda_j^2)^{a_j}\right]^{\rho/2}}{\sum_1^n (1+\lambda_j^2)^{r_j}} \tilde g = \frac{1}{\left[\sum_1^n (1+\lambda_j^2)^{a_j}\right]^{\rho/2}} \tilde h, \qquad (2)$$

where $a_j = r_j/\rho$, $\rho = \max_j r_j$. According to theorems on multipliers (^{2,4,5}), from (2) we obtain that the function $h \in L_p$. Equality (2) means that the function f is represented by the convolution

$$f = G_r * h, \qquad h \in L_p, \qquad (3)$$

where G_r is the Fourier transform of the function $\left[\sum (1+\lambda_j^2)^{a_j}\right]^{-\rho/2}$. We shall see below,

that the function G_r is summable and, consequently, the convolution (3) is written in the form

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{E_n} G_r(x-y) h(y)\, dy. \qquad (4)$$

We shall call the integral on the right the r-Bessel potential of the function h. Isotropic Bessel potentials ($r_1 = r_2 = \dots = r_n$) of functions $h \in L_p$ have been studied in detail by many authors (see, for example, Aronszajn' s survey (^{6})). The results of the work (^{2}) are adjoined by

Theorem 1. *The totality of r-Bessel potentials of functions from L_p coincides with the space $L_p^{(r)}$, i.e., for each function $f \in L_p^{(r)}$ one can indicate a function $h \in L_p$ such that f is represented in the form of the r-Bessel potential of h, and moreover $\|h\|_{L_p} \le c\|f\|_{L_p^{(r)}}$; conversely, the r-Bessel potential of a function $h \in L_p$ belongs to $L_p^{(r)}$, and $\|f\|_{L_p^{(r)}} \le c\|h\|_{L_p}$.*

Using the representation (4) and estimates of the kernel G_r, we obtain an embedding theorem for the spaces $L_p^{(r)}$, $r_j > 0$, $j = 1, \dots, n$.

Theorem 2. *Let $1 < p < p' < \infty$, $1 \le m \le n$, $\mathbf{k} = (k_1, \dots, k_n)$,*

$$k = \sum_{j=1}^n k_j, \qquad \chi(k) = 1 - \left(\frac{1}{p} - \frac{1}{p'}\right) \sum_1^m \frac{1}{r_j} - \frac{1}{p} \sum_{m+1}^n \frac{1}{r_j} - \sum_1^n \frac{k_j}{r_j} \ge 0.$$

Then, if $f(x_1, \dots, x_n) \in L_p^{(r_1,\dots,r_n)}(E_n)$, then the function

$$\psi(x_1, \dots, x_m, x_{m+1}^0, \dots, x_n^0) = \frac{\partial^k f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} (x_1, \dots, x_m, x_{m+1}^0, \dots, x_n^0)$$

for arbitrary fixed $x^0_{m+1}, \dots, x^0_n$ *belongs to the space* $L^{(\rho_1,\dots,\rho_m)}_{p'}(E_m)$, $\rho_j = \chi(k) r_j$, $j = 1, \dots, n$. *Moreover, the estimate holds*

$$\|\psi\|_{L^{(\vec{\rho})}_{p'}(E_m)} \le c\|f\|_{L^{(r)}_p(E_n)},$$

where c *does not depend on* f.

The following theorem generalizes the theorem of V. I. Kondrashov (on the complete continuity of the embedding operator into c, see (1), p. 91).

Theorem 3. *Let* $1 < p < \infty$, $\chi_\infty = 1 - \frac{1}{p}\sum_1^n \frac{1}{r_j} > 0$, $\rho_j = \chi_\infty r_j$, $j = 1, \dots, n$. *Then*

$$L^{(r_1,\dots,r_n)}_p(E_n) \to H^{(\rho_1,\dots,\rho_n)}_\infty(E_n),$$

i.e., from $f \in L^{(r_1,\dots,r_n)}_p$ *it follows that* $f \in H^{(\rho_1,\dots,\rho_n)}_\infty$, *and for some constant* c, *independent of* f,

$$\|f\|_{H^{(\rho_1,\dots,\rho_n)}_\infty} \le c\|f\|_{L^{(r_1,\dots,r_n)}_p}.$$

Here $H^{(\rho_1,\dots,\rho_n)}_\infty(E_n) = H^{(\vec{\rho})}_\infty$ are classes of functions satisfying the corresponding Hölder condition (see (7)). Theorem 3 follows from the embeddings

$$L^{(r)}_p \to B^{(r)}_{p,\theta}, \quad \max(p,2) \le \theta \ (^8); \qquad B^{(r)}_{p,\theta} \to H^{(r)}_p \to H^{(\vec{\rho})}_\infty \ (^9).$$

If $\chi(0) > 0$, then for $p \ge 2$ the embedding

$$L^{(r)}_p(E_n) \to B^{(\vec{\rho})}_p(E_m),$$

holds, and for $1 < p \le 2$ the embedding

$$L^{(r)}_p(E_n) \leftarrow B^{(\vec{\rho})}_p(E_m);$$

this follows easily from the results of (8,9). Moreover, the boundary values of functions from $L^{(r)}_p$ are completely characterized (as in the isotropic case (10)) in terms of $B^{(\vec{\rho})}_p$-classes, i.e., there holds

Theorem 4. $L^{(\mathbf{r})}_p(E_n) \leftrightarrow \overline{B}^{(\rho)}_p(E_m)$, $\rho_j = \varkappa(0) r_j$, $j = 1, \dots, m$, $\varkappa(0) > 0$, $r_j > 0$, $1 < p < \infty$.

2. The proof of Theorem 2 is carried out according to the plan of [2]. The estimates of the kernel $G_r(x)$ needed for this will now be derived. Formally, $G_r(x)$ is written in the form

$$G_r(x) = \frac{1}{2\pi^{n/2}} \int_{E_n} \frac{e^{i\lambda x}\, d\lambda}{\left[\sum_1^n (1+\lambda_j^2)^{\alpha_j}\right]^{\rho/2}}, \tag{5}$$

where $\alpha_j > 0$ are fixed, while ρ is regarded as a parameter. The written integral converges absolutely for $\rho > \sum_1^n \frac{1}{\alpha_j}$, and $G_r(x)$ is in this case a summable function (as will be seen from the subsequent estimates); therefore, by Parseval's equality, we have

$$\frac{1}{(2\pi)^{n/2}} \left(\int \frac{e^{i\lambda x}\, d\lambda}{[R(\lambda)]^{\rho/2}}, \varphi \right) = \left(R^{-\rho/2}(\lambda), \widetilde{\varphi} \right), \quad R(\lambda) = \sum_1^n (1+\lambda_j^2)^{\alpha_j}, \quad \varphi \in S. \tag{6}$$

The left-hand side of this equality is an analytic function of the parameter ρ, and at the same time, for all ρ, it is equal to $\langle G_r, \varphi \rangle$ (by the definition of the Fourier transform). Consequently, the function $G_r(x)$ in (5), for small ρ, should be understood as the analytic continuation with respect to ρ. Let $x' = (x_1, \dots, \dots, x_{n-1})$, $\lambda' = (\lambda_1, \dots, \lambda_{n-1})$; write

$$(2\pi)^{n/2} G_r(x) = \int_{E_{n-1}} e^{i\lambda' x'}\, d\lambda' \int_{-\infty}^{\infty} \frac{e^{i x_n \lambda_n}}{R^{\rho/2}(\lambda)}\, d\lambda_n. \tag{7}$$

Using the analyticity of the integrand in the inner integral, we deform the contour of integration and pass from integration along the real axis to integration along the line of the plane $v_n = \lambda_n + i\mu_n$ (a similar device is used in (11)).

Lemma. *There exist positive constants k, b such that, for*

$$0 \le \mu_n \le k \sum_1^n (1+\lambda_j^2)^{\alpha_j/2\alpha_n}$$

the inequality

$$R(v) \ge b \sum_1^n (1+\lambda_j^2)^{\alpha_j}$$

holds, where $v = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n + i\mu_n)$.

Thus, if as L we choose $L:\ v_n = \lambda_n + i\mu_n$,

$$\mu_n = k \sum_1^n (1+\lambda_j^2)^{\alpha_j/2\alpha_n},$$

then the inner integral in (7) does not change, and we obtain

$$(2\pi)^{n/2} G_r(x) = \int e^{ix'\lambda'}\, d\lambda' \int_L \frac{e^{ix_{nv}n}}{R^{\rho/2}(\lambda)}\, dv_n. \tag{8}$$

The integral on the right has meaning for all $\rho > 0$ and represents the desired analytic continuation; we shall use it for estimates. Note that we take $x_n > 0$, since from (5) it is seen that $G_r(x)$ depends only on $|x_j|,\ j = 1, \dots, n$. We have

$$(2\pi)^{n/2} |G_r(x)| \le c \int_{E_n} \frac{\exp\left[-k x_n \sum_1^n (1+\lambda_j^2)^{\alpha_j/2\alpha_n}\right]\, d\lambda}{\left\{\sum_1^n (1+\lambda_j^2)^{\alpha_j/2\alpha_n}\right\}^{\rho\alpha_n}} = c\left\{\int_Q + \int_{E_n - Q}\right\}.$$

The integral over $Q = \{\lambda;\ |\lambda_j| \le 1,\ j = 1, \dots, n\}$ is majorized by the quantity e^{-knx_n}. Let us estimate the integral over $E_n - Q$:

$$\begin{aligned} e^{k_1 n x_n} \int_{E_n - Q} &= \int_{E_n - Q} \frac{\exp\left[-k x_n \sum_1^n |\lambda_j|^{\alpha_j/\alpha_n}\right]}{\left(\sum_1^n |\lambda_j|^{\alpha_j/\alpha_n}\right)^{\alpha_n \rho}}\, d\lambda \le \\ &\le c \int_1^\infty \frac{\exp\left[-\frac{k}{\sqrt{n}} x_n r\right]\, dr}{r^{\alpha_n\rho - \sum \alpha_n/\alpha_j + n}} = c \int_1^\infty \frac{\exp\left[-\frac{k}{\sqrt{n}} x_n t\right]\, dt}{t^{1 - r_n(\sum 1/r_j - 1)}}. \end{aligned} \tag{9}$$

Further estimation presents no difficulty. Similar computations may be carried out by replacing x_n by $x_j,\ j = 1, \dots, n-1$. In addition, if one differentiates expression (8), then we obtain

$$(2\pi)^{n/2} \frac{\partial^l G_r(x)}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} = \int_{E_{n-1}} (i\lambda_1)^{l_1} \cdots (i\lambda_{n-1})^{l_{n-1}} e^{i\lambda' x'}\, d\lambda \int_L \frac{(i\nu_n)^{l_n} e^{ix_n\nu_n}\, d\nu_n}{R^{\rho/2}(\nu)}.$$

Estimates of this derivative, entirely analogous to those just given, lead to an integral of type (9) with the sum $\sum_1^n 1/r_j$ replaced by $\sum_1^n (1+l_j)/r_j$.

The final estimates have the following form:

Theorem 5. Let the function $G_r(x)$, $r = (r_1, \dots, r_n)$, $r_j > 0$, be defined by formula (5) for

$$\rho > \sum_1^n \frac{1}{\alpha_j}, \qquad \alpha_j = r_j/\rho, \qquad \rho = \max_j r_j, \qquad j = 1, \dots, n,$$

and for each $x \ne 0$ analytically continued in ρ to positive values of ρ. The function $G_r(x)$ is infinitely differentiable for $x \ne 0$ and satisfies the estimates (for $k = 1, \dots, n$)

$$\left|\frac{\partial^l G_r(x)}{\partial x_1^{l_1}\dots\partial x_n^{l_n}}\right| \le \begin{cases} \dfrac{c}{|x_k|^{r_k(\sum(1+l_j)/r_j-1)}}, & \text{if } \displaystyle\sum_{j=1}^{n}\frac{1+l_j}{r_j} > 1, \\ c\ln\left(1+\dfrac{1}{|x_k|}\right), & \text{if } \displaystyle\sum_{j=1}^{n}\frac{1+l_j}{r_j} = 1, \\ c, & \text{if } \displaystyle\sum_{j=1}^{n}\frac{1+l_j}{r_j} < 1. \end{cases} \tag{10}$$

Moreover, for large $|x|$ the estimate

$$\left|\partial^l G_r(x)/\partial x_1^{l_1}\dots\partial x_n^{l_n}\right| \le ce^{-a|x|} \tag{11}$$

is valid, where the constants $c > 0,\ a > 0$ depend only on r, l, n.

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[*] k_1 is a certain constant, $0 < k_1 \leq k$.

Note: Figure translations are in progress. See original paper for figures.