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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE REPRESENTATION OF CUSP FORMS BY THETA SERIES

*(Presented by Academician Yu. V. Linnik on 20 V 1965)*

Hecke <sup>(1)</sup> expressed the conjecture that all integral modular forms of type  $\{\Gamma_0(p), -2\}$  and character  $\chi = 1$  ( $p$  a prime number) can be expressed as linear combinations of quaternary theta series. This conjecture was proved by Eichler <sup>(2)</sup>, who obtained, moreover, more general results <sup>(2,3)</sup>. We shall restrict ourselves to the case of cusp forms. Eichler's method consists in comparing representations of the modular correspondences  $T_n$  in the space of cusp forms and in its subspace generated by theta series.

The purpose of the present note is to supplement Eichler's results (in the case of prime level  $p$ ) by establishing the dependence of the coefficients in the expansion of cusp forms that are eigenfunctions of the Hecke operators on the level  $p$ . Estimates of power type (with respect to  $p$ ) are obtained. These results will probably be useful in some questions of number theory.

**Theorem 1.** Let

$$\varphi_i(\tau) = \sum_{n=1}^{\infty} \tau_i(n) e^{2\pi i n \tau}, \quad \tau_i(1) = 1, \quad i = 1, 2, \dots, g,$$

be a basis of the space  $\mathfrak{S}$  of cusp forms of type  $\{\Gamma_0(p), -2\}$  and character  $\chi = 1$ , consisting of eigenfunctions of all Hecke operators  $T_n$ ,  $(n, p) = 1$ , acting on this space, and orthogonal <sup>(4)</sup>. Then

$$\varphi_i(\tau) = \sum_{t=1}^{g^2} \beta_t^{(i)} \vartheta'_t(\tau), \quad i = 1, 2, \dots, g, \quad (1)$$

where  $\vartheta'_t(\tau)$  are cusp forms that are differences of quaternary theta series of level  $p$  (discriminant  $p^2$ ), and

$$|\beta_t^{(i)}| < C_\varepsilon p^{5+\varepsilon}. \quad (2)$$

**Proof.** Result (1) was proved by Eichler in <sup>(2)</sup>. Consider the representation  $R_1(n)$  of the ring of operators  $T_n$ ,  $(n, p) = 1$ , in the space  $\mathfrak{S}$ :

$$\Phi \circ T_n = R_1(n)\Phi,$$

where

$$\Phi = \begin{pmatrix} \varphi_1(\tau) \\ \vdots \\ \varphi_g(\tau) \end{pmatrix}.$$

The trace of this representation was computed in <sup>(2,5,6)</sup>.

Let  $Q$  be the algebra of definite quaternions of discriminant  $p^2$ ;  $\mathfrak{D}_1$  an order in  $Q$  of level  $p$ ;  $\mathfrak{M}_\nu$ , ( $\nu = 1, \dots, H$ ), a system of representatives of the classes of left  $\mathfrak{D}_1$ -ideals;  $\mathfrak{D}_\nu$  the right orders of the ideals  $\mathfrak{M}_\nu$ ;  $\mathfrak{M}_\mu^{-1}\mathfrak{M}_\nu$  runs through a system of representatives of all classes of left  $\mathfrak{D}_\mu$ -ideals. Let  $\pi_{\mu\nu}(n)$  be the number of all left-equivalent with  $\mathfrak{M}_\mu^{-1}\mathfrak{M}_\nu$  integral ideals of norm  $n$ , and  $w_\nu$  the number of units in  $\mathfrak{D}_\nu$ . Consider  $P(n) = (\pi_{\mu\nu}(n))$ , a matrix of order  $H$  (Anzahlmatrix 2; ideal number matrix <sup>(7)</sup>);

$$P(0) = \begin{pmatrix} w_1^{-1} & \dots & w_H^{-1} \\ \cdot & \cdot & \cdot \\ w_1^{-1} & \dots & w_H^{-1} \end{pmatrix}.$$

Let  $\vartheta(\tau)$  be the matrix whose elements are quaternion theta series:

$$\vartheta(\tau) = \sum_{n=0}^{\infty} P(n)e^{2\pi in\tau}.$$

We have (see (7))

$$M^{-1}P(n)M = \begin{pmatrix} d_n & p(n) \\ 0 & P'(n) \end{pmatrix};$$

$$\vartheta'(\tau) = \sum_{n=1}^{\infty} P'(n)e^{2\pi in\tau} = (\vartheta'_{\mu\nu}(\tau))$$

is a matrix of order  $H - 1$ , whose elements are differences of quaternion theta series of degree  $p$ .

Eichler considered the representation of the ring of operators  $T_n$ ,  $(n, p) = 1$ , in the subspace  $\Theta$  of the space  $\mathfrak{S}$  generated by parabolic forms—differences of theta series, proving that

$$\vartheta'(\tau) \circ T_n = P'(n)\vartheta'(\tau), \quad (n, p) = 1,$$

and found (8) the trace of the representation  $P'(n)$ . It turned out that  $\text{Sp } P'(n) = \text{Sp } R_1(n)$ ,  $(n, p) = 1$ . From the complete reducibility of both representations and from the equality of their traces there follows the equivalence of

these representations. Hence (taking into account one remark of Hecke (1)) we derive the equality

$$A\Phi'(\tau)A^{-1} = \vartheta'(\tau), \quad (3)$$

where  $\Phi'(\tau)$  is a diagonal matrix of the form  $[\varphi_1(\tau), \dots, \varphi_g(\tau)]$ , and  $A$  is a regular matrix of order  $g$ .

Multiply the matrices on the left and equate the corresponding elements of both sides. We obtain

$$\vartheta'_{\mu\nu}(\tau) = \sum_{i=1}^g \alpha_i^{(\mu\nu)} \varphi_i(\tau).$$

Squaring both sides of the equality (with respect to the Petersson metric), and taking into account the orthogonality of the basis  $\{\varphi_i(\tau)\}$ ,  $i = 1, \dots, g$ , we have

$$(\vartheta'_{\mu\nu}(\tau), \vartheta'_{\mu\nu}(\tau)) = \sum_{i=1}^g |\alpha_i^{(\mu\nu)}|^2 (\varphi_i(\tau), \varphi_i(\tau)).$$

Using the methods of paper (9), one can obtain the estimates

$$(\varphi_i(\tau), \varphi_i(\tau)) > Cp^2;$$

$$(\vartheta'_{\mu\nu}(\tau), \vartheta'_{\mu\nu}(\tau)) < C_\varepsilon p^{9+\varepsilon}.$$

Hence follows the estimate  $|\alpha_i^{(\mu\nu)}| < C_\varepsilon p^{5+\varepsilon}$ . On the other hand, taking equality (3) in the form

$$\Phi'(\tau) = A^{-1}\vartheta'(\tau)A$$

and again equating the corresponding elements of both sides, we obtain

$$\varphi_i(\tau) = \sum_{t=1}^{g^2} \beta_t^{(i)} \vartheta'_t(\tau), \quad i = 1, \dots, g$$

(for simplicity we have introduced a new numbering of the elements of the matrix  $\vartheta'(\tau)$ ).

It remains to note that each of the coefficients  $\beta_t^{(i)}$  coincides with some  $a_j^{(\mu\nu)}$ . The theorem is proved.

The proof of the more general Theorem 2 is constructed on the same idea; however, it is technically very complicated. This theorem supplements Eichler's result from (3).

**Theorem 2.** *Every eigenfunction (of all Hecke operators  $T_n$ ,  $(n, p) = 1$ )*

$$\varphi_i(\tau) = \sum_{n=1}^{\infty} \tau_i(n) e^{2\pi i n \tau}, \quad \tau_i(1) = 1,$$

*from an orthogonal basis of the space of parabolic forms of type  $\{\Gamma_0(p), -2k\}$  and character  $\chi = 1$  can be expressed in the form of a linear combination of quaternary theta series of degree  $p$  with homogeneous Laplace spherical functions<sup>(10)</sup>, and for the coefficients of the expansion the estimate holds*

$$|\beta_t| < C_\varepsilon p^{5/2k+4+\varepsilon}. \quad (4)$$

In the proof, in addition to Eichler's results, we use a number of results obtained in (10,11).

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*Note: Figure translations are in progress. See original paper for figures.*

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