STUDY OF A NEIGHBORHOOD OF A SINGULAR POINT OF A MULTIDIMENSIONAL DIFFERENTIAL EQUATION IN THE ANALYTIC CASE

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Abstract

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STUDY OF A NEIGHBORHOOD OF A SINGULAR POINT OF A MULTIDIMENSIONAL DIFFERENTIAL EQUATION IN THE ANALYTIC CASE

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1. Let E_x be a finite-dimensional real vector space, and E_y a finite-dimensional vector space (real or complex). Consider the completely solvable multidimensional differential equation of first order

$$y'(x)h = \sum_{j=1}^{\infty} A_j h y(x) \dots y(x), \tag{1}$$

where $A_j h y_1 \dots y_j$ is a $(j+1)$-linear operator defined on

$$E_x \oplus \underbrace{E_y \oplus \dots \oplus E_y}_{j}$$

with values in the space E_y, symmetric in the variables $y_1, \dots, y_j$. For the case when the space E_x is one-dimensional, in the investigations of Poincaré (^{1}), Picard (^{2}), and Dulac (^{3}) (see also (^{4,5})), conditions were found under which, by the change of variables

$$z = \sum_{j=1}^{\infty} C_j y^j, \qquad C_1 = 1 \tag{2}$$

(C_j is a j-linear symmetric operator defined on

$$\underbrace{E_y \oplus \dots \oplus E_y}_{j}$$

with values in E_y), equation (1) is transformed into the differential equation of first approximation

$$z'(x)h = A_1hz(x). \tag{3}$$

An attempt to carry over the results of these investigations to the multidimensional case was made in (^{6}), where $\dim E_x = 2$ and $\dim E_y = 3$. The author takes this opportunity to express gratitude to A. D. Myshkis for pointing out this work. In the present article we consider the general case.

2. The conditions of complete solvability of the differential equation (1) can be written in the form (see, for example, (^{7}))

$$\Lambda_{hk}\left[\sum_{j=1}^{p} V_{y_1\cdots y_p}\left(A_jhy_1\ldots y_{j-1}A_{p-j+1}ky_j\ldots y_p\right)\right] = 0, \qquad p = 1, 2, \ldots \tag{4}$$

Here Λ and V are the operators of taking the skew-symmetric and symmetric parts with respect to the corresponding variables. For $p = 1$ we obtain

$$A_1hA_1k = A_1kA_1h. \tag{5}$$

Operators satisfying the last condition were studied in (^{8}). Let $\lambda^1, \lambda^2, \ldots, \lambda^n$ be the proper functionals of the operator A_1, each of them repeated as many times as its multiplicity. Recall that a linear functional λ on E_x (real or complex-

$\cdots$) is called an eigenvalue if there exists a vector $f \in E_y$ such that for all $h \in E_x$ one has $Ahf = (\lambda h)f$.

We shall say that the **Lyapunov condition** is satisfied if

$$\lambda^j \neq \sum_{s=1}^{n} m_s\lambda^s, \qquad \text{where } m_s \geq 0 \text{ and } \sum_{s=1}^{n} m_s > 1. \tag{6}$$

We shall say that the **Poincaré condition** is satisfied if the convex hull of the set $\lambda^1, \ldots, \lambda^n$ does not contain zero.

Let

$$\sigma_p = \min_{\substack{j=1,2,\ldots,n \\ m_s \geq 0,\ \sum m_s = p}} \left\| \lambda^j - \sum_{s=1}^{n} m_s\lambda^s \right\|. \tag{7}$$

If the Lyapunov and Poincaré conditions are satisfied, then one can indicate a positive number ε such that

$$\sigma_p \geq \varepsilon(p-1) \qquad (p = 2, 3, \ldots). \tag{8}$$

(The spaces E_x and E_y are assumed to be normed; therefore the spaces E_x^* (of all real functionals) and $\bar{E}_x^*$ (of all complex functionals) are also normed.)

3. Let us consider the formal differential equation (1). The formal power series (2) transforms equation (1) into equation (3) if and only if its coefficients C satisfy the following conditions:

$$A_1 h C_p y_1 \dots y_p - C_p (A_1 h y_1) \dots y_p - \dots - C_p y_1 \dots (A h y_p) = B_p h y_1 \dots y_p, \tag{9}$$

where

$$B_p h y_1 \dots y_p = \sum_{j=1}^{p-1} j V^j_{y_1 \cdots y_p} C_j y_1 \dots y_{j-1} A_{p-j+1} h y_j \dots y_p. \tag{10}$$

Let us consider an equation of the form

$$A_1 h C y_1 \dots y_p - C (A_1 h y_1) \dots y_p - \dots - C_p y_1 \dots (A h y_p) = B h y_1 \dots y_p. \tag{11}$$

Here $B h y_1 \dots y_p$ is a given $(p+1)$-linear operator, symmetric in the variables $y_1, \dots, y_p$, and the symmetric operator $C y_1 \dots y_p$ is the unknown; $p \geq 2$.

Lemma 1. *Under the Lyapunov condition, equation (11) has a (unique) solution if and only if*

$$\Lambda_h k (A_1 h B k y_1 \dots y_p - B k (A_1 h y_1) \dots y_p - \dots - B k y_1 \dots (A_1 h y_p)) = 0. \tag{12}$$

The proof of this lemma is based on the results of the paper (^{8}).

We now give the first main result of the present paper.

Theorem 1. *Suppose that the condition of complete resolvability (4) and the Lyapunov condition are satisfied. Then there exists a unique formal power series (2) that transforms the formal multidimensional first-order differential equation (1) into the differential equation (3).*

The proof of this theorem is rather laborious. We outline its main stages. Using the conditions of complete resolvability, it is comparatively simple to prove the unique solvability of equations (9) for $p = 2$ and for $p = 3$.

In what follows we use the method of mathematical induction. For

for $p > 3$ the solvability condition for equation (9) can be brought to the form

$$\begin{aligned}&\sum_{j=1}^{p-2} j\Lambda_{hk}V_{y_1\dots y_p}\left\{\sum_{s=j+1}^{p-1}(s-j+1)Cjy_1\dots y_{j-1}Vy_j\dots y_p\left[A_{s-j+1}ky_j\dots\right.\right.\\&\qquad\left.\left.\dots y_{s-1}A_{p-s+1}hy_s\dots y_p+A_{s-j+1}hy_j\dots y_{s-1}A_{p-s+1}ky_s\dots y_p\right]\right\}+\\&+\sum_{j=2}^{p-2} j\Lambda_{hk}V_{y_1\dots y_p}\left\{\sum_{s=j+1}^{p-1}(j-1)\widetilde{V}^{j}_{y_1\dots\bar{y}_s}Cjy_1\dots y_{j-1}A_{s-j+1}hy_j\dots\right.\\&\qquad\left.\dots y_{s-1}A_{p-s+1}ky_s\dots y_p\right\}=0.\end{aligned}\tag{13}$$

The first expression is equal to zero by virtue of the properties of the operators Λ and V. The second expression is equal to zero, since the indices of the operators A run through one and the same interval $2 \le t \le p-j$, and the operators C are symmetric (the notation $\widetilde{V}^{j}_{y_1\dots\bar{y}_s}$ means that first symmetrization has been performed with respect to $y_1, \dots, \bar{y}_s$, with only those permutations σ taken into account for which $1 \le \sigma(s) \le j-1$, and then $\bar{y}_s = A_{p-s+1}ky_s\dots y_p$ is put).

4. In the n-dimensional space E_y take some basis $f_1, f_2, \dots, f_n$ and introduce the norms of the p-linear operator C and the $(p+1)$-linear operator B as follows:

$$\|C\|_* = \max_{1\le j\le n}\sum_{\substack{1\le j_s\le n\\ s=1,2,\dots,p}}\left|C^{j}_{j_1\dots j_p}\right|,\tag{14}$$

$$\|B\|_* = \max_{1\le j\le n}\sum_{\substack{1\le j_s\le n\\ s=1,2,\dots,p}}\left\|B()^{j}_{j_1\dots j_p}\right\|.\tag{15}$$

Here $C^{j}_{j_1\dots j_p} = (Cf_{j_1}\dots f_{j_p})^j$, while $B()^{j}_{j_1\dots j_p}$ is the linear functional $(Bhf_{j_1}\dots f_{j_p})^j$.

Lemma 2. If the operator B has the form

$$Bhy_1\dots y_p = \sum_{j=1}^{p-1} jV_{y_1\dots y_p}Cjy_1\dots y_{j-1}A_{p-j+1}hy_j\dots y_p,\tag{16}$$

then the inequality

$$\|B\|_* \le \sum_{j=1}^{p-1} j\|C_j\|_*\|A_{p-j+1}\|_*\tag{17}$$

holds.

Lemma 3. Let the numerical power series $\varphi(u) = \sum_{j=1}^{\infty} a_{ju}^{j}$ have a nonzero radius of convergence and $a_1 \neq 0$. Then there exists a unique analytic solution of the differential equation $f'(u)\varphi(u) = a_1 f(u)$, satisfying the initial data $f(0) = 0,\ f'(0) = 1$.

5. In this section we shall show that transformation (2) is analytic in some neighborhood of the origin of coordinates. In doing so we use the theory of analytic functions, developed in the complex case in (9) and in the real case in (10).

We shall say that the operator A_1 has a **simple structure** if in the space E_y one can specify a basis consisting of eigenvectors of the operator A_1.

The second principal result of the present article is as follows.

Theorem 2. Consider the multidimensional differential equation (1). Suppose that the operator A_1 has a simple structure and $f_1, f_2, \dots, f_n$ is a basis composed of eigenvectors of the operator A_1. Introduce the norm $\| \ \|_*$ (see item 4).

Suppose that the power series

$$\sum_{j=1}^{\infty} \|A_j\|_* u^j$$

has a nonzero radius of convergence (and, consequently,

$$g(y)h = \sum_{j=1}^{\infty} A_j h y^j$$

for any $h \in E_x$ is analytic in some ball U).

Suppose that the Lyapunov and Poincaré conditions are satisfied.

Then there exists a unique analytic transformation of the form (2), which, in some neighborhood of the zero singular point of the space E_y, transforms the multidimensional differential equation (1) into the multidimensional differential equation (3).

(Here, of course, it is assumed that the conditions of complete solvability (4) of the multidimensional differential equation (1) are satisfied.)

The proof is carried out by the method of majorants, using Lemmas 1-3. The principal role is played by the inequality

$$\|C_p\|_* \leq \frac{1}{\varepsilon(p-1)} \sum_{j=1}^{p-1} j \|C_j\|_* \|A_{p-j+1}\|_*, \tag{18}$$

where the constant ε is the same as in inequality (8).

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Note: Figure translations are in progress. See original paper for figures.