ON THE SIGNATURES OF TOEPLITZ FORMS

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Abstract

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MATHEMATICS

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ON THE SIGNATURES OF TOEPLITZ FORMS

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1. Let a Hermitian form be given

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\bar{\xi}_k$$

of order n, rank ρ, and signature σ: $\rho = \pi + \nu$, $\sigma = \pi - \nu$, where π is the number of positive and ν the number of negative squares of the given form. By $D_q = \det |a_{jk}|_{j,k=1}^{q}$ $(q = 1, 2, \dots, n)$ we shall denote the successive leading minors of the matrix $\|a_{jk}\|_{j,k=1}^{n}$.

The well-known Jacobi rule (see, for example, ([1]), Ch. X) makes it possible to find the numbers π and ν from the formulas

$$\pi = P(1;\ D_1,\ D_2, \dots, D_\rho), \qquad \nu = V(1;\ D_1,\ D_2, \dots, D_\rho), \tag{1}$$

where the symbols P and V denote, respectively, the numbers of permanences of sign and changes of sign in the sequence of numbers indicated in parentheses to the right of these symbols.

Jacobi' s rule, as is clear from its very formulation, has meaning only under the condition that $D_q \neq 0$ $(q = 1, 2, \dots, \rho)$. Gundelfinger and Frobenius gave generalizations of this rule to the case when $D_\rho \neq 0$, and in the sequence of numbers $1, D_1, D_2, \dots, D_{\rho-1}$ there are zeros, but not two (respectively three) consecutive zeros ([1,2]).

In the special case of a real Hankel form

$$\sum_{p,q=0}^{n-1} s_{p+q}\xi_p\xi_q$$

Frobenius ([2]) freed Jacobi' s rule from all the above-listed restrictions, indicating what signs should be assigned to the zero minors in the sequence $1, D_1, D_2, \dots, D_\rho$ so that formulas (1) would remain valid.* The present note establishes an analogous result for Hermitian Toeplitz forms

$$\sum_{p,q=0}^{n-1} c_{p-q}\xi_p\overline{\xi}_q \qquad (c_{-p} = \overline{c}_p;\ p = 0, 1, \dots, n-1). \tag{2}$$

At the same time, certain specific facts are revealed (Proposition 4°, Lemmas 1 and 2) that have no analogues in the theory of Hankel forms. Meanwhile, the main result (Theorem 1) is entirely analogous to Frobenius' rule. In the concluding § 4 a complete description is given of one class of finite Toeplitz matrices that has remained unstudied up to now.

2. Slightly modifying the notation introduced above in the case that interests us, put

$$\Delta_j = \det |c_{p-q}|_{p,q=0}^{j} \qquad (j = 0, 1, 2, \dots, n-1),$$

where $\|c_{p-q}\|_{p,q=0}^{n-1}$ is the matrix of the form (2). In addition, set $\Delta_{-1} = 1$.

We shall start from the following results, which partially entered into our recent note ([3]).

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* A proof of Frobenius' rule in a more modern exposition may be found in ([1]), Ch. X, Theorems 23 and 24. This exposition should be supplemented by the reasoning contained in the original paper ([2]), showing that the indicated theorems remain valid also in the case when all $D_q = 0$ $(q = 1, 2, \dots, \rho)$ (a case not considered in ([1]), but actually possible).

1°. If the form (2) has rank $(0 <)\rho < n$, and the "shortened" form

$$\sum_{p,q=0}^{n-2} c_{p-q}\xi_p\overline{\xi}_q$$

has rank $\rho' < \rho$ and determinant $\Delta_{\rho'-1} \neq 0$, then $\rho' = \rho - 2$, and the signatures of both forms coincide*.

2°. If the rank of the form (2) is equal to $\rho(> 0)$ and $\Delta_{\rho-1} = 0$, then for any continuation of the sequence of coefficients $\{c_p\}_0^{n-1}$ by an arbitrary element $c_n(= \overline{c}_{-n})$, the rank of the "continued" form

$$\sum_{p,q=0}^{n} c_{p-q}\xi_p\overline{\xi}_q$$

is equal to $\rho_1 = \rho + 2$, and the signatures of both forms coincide*.

3°. Under the conditions of assertion 2°, the conclusion of assertion 1° is also valid.

4°. Under the conditions of assertion 2°, define a nonnegative integer r by the condition

$$\Delta_{\rho-1} = \Delta_{\rho-2} = \ldots = \Delta_r = 0, \qquad \Delta_{r-1} \neq 0.$$

Then $\rho = r + 2k$ (integer $k > 0$), and the matrix $\|c_{p-q}\|_{p,q=0}^{n-1}$ has a nonzero principal minor $\widehat{\Delta}_{\rho-1}$ of order ρ, formed, when $r = 0$, from the first k and the last k lines (i.e., columns and rows) of this matrix, and when $r \geq 1$, by bordering the minor Δ_{r-1} with the nearest k lines to it and the last k lines of the same matrix.

Let us note that assertions 3° and 4° are derived from assertion 2°—the theorem which formed the main content of work (3).

3. Consider the series of numbers

$$(1 =)\Delta_{-1}, \Delta_0, \Delta_1, \ldots, \Delta_{\rho-1},$$

where $\rho(> 0)$ is the rank of the Hermitian matrix $\|c_{p-q}\|_{p,q=0}^{n-1}$, and suppose that in this series there is a group of numbers

$$\Delta_{h-1} \neq 0, \qquad \Delta_h = \Delta_{h+1} = \ldots = \Delta_{h+m-1} = 0, \qquad \Delta_{h+m} \neq 0$$

$$(0 \leq h \leq \rho - 2; \qquad 1 \leq m \leq \rho - h - 1). \tag{3}$$

From assertions 1°-3° there follows the following result (which has no analogue in the theory of Hankel forms).

Lemma 1. The number m is always odd: $m = 2l + 1$. Moreover, if l is even, then $\operatorname{sign} \Delta_{h+m} = -\operatorname{sign} \Delta_{h-1}$, and if l is odd, then $\operatorname{sign} \Delta_{h+m} = \operatorname{sign} \Delta_{h-1}$.

Under the assumptions (3), consider the forms

$$\sum_{p,q=0}^{h} c_{p-q} \xi_p \bar{\xi}_q, \tag{A}$$

$$\sum_{p,q=0}^{h+m} c_{p-q} \xi_p \bar{\xi}_q. \tag{B}$$

Lemma 2. The number of positive and the number of negative squares of the form (B) are each greater than the corresponding numbers for the form (A) by $l+1$ units ($l=(m-1)/2$).

Lemmas 1 and 2 already make it possible to formulate the rule of interest to us in the case when $\Delta_{\rho-1}\neq 0$. If, however, $\Delta_{\rho-1}=0$, then the following comes to our aid.

Lemma 3. Let $\widehat{\Delta}_{\rho-1}(\neq 0)$ be the principal minor of the matrix $\|c_{p-q}\|_{p,q=0}^{n-1}$, whose existence was established in assertion 4°, and let r,k be the numbers defined there ($\rho=r+2k, \Delta_{r-1}\neq 0$). Then $\operatorname{sign}\widehat{\Delta}_{\rho-1}=-\operatorname{sign}\Delta_{r-1}$ when k is odd, and $\operatorname{sign}\widehat{\Delta}_{\rho-1}=\operatorname{sign}\Delta_{r-1}$ if k is even.

* It is not difficult to verify that the assertion about signatures follows from the assertion about ranks.

The assertion of Lemma 3 is easily verified with the aid of the calculational lemma from ([3]).

Theorem 1 (Generalization of Jacobi' s rule to the case of an arbitrary Toeplitz form). *Let, for the form (2) of rank $\rho(>0)$, the sequence*

$$(1=)\Delta_{-1},\Delta_0,\Delta_1,\dots,\Delta_{\rho-2},\Delta_{\rho-1}, \tag{4}$$

be formed, in which $\widetilde{\Delta}_{\rho-1}=\Delta_{\rho-1}$ if $\Delta_{\rho-1}\neq 0$, while for $\Delta_{\rho-1}=0$ the minor $\widetilde{\Delta}_{\rho-1}$ is defined according to Proposition 4^0. If there are groups of zeros of the form (3) in the sequence (4), signs are assigned to them by the rule

$$\operatorname{sign}\Delta_{h-1+j}=(-1)^{j(j+1)/2}\operatorname{sign}\Delta_{h-1}\quad (j=1,2,\dots,m).$$

Then the signature $\sigma=\pi-\nu$ of the form (2) and the numbers π,ν are determined by the formulas:

$$\pi=P(1,\Delta_0,\Delta_1,\dots,\Delta_{\rho-2},\widetilde{\Delta}_{\rho-1});\quad \nu=V(1,\Delta_0,\Delta_1,\dots,\Delta_{\rho-2},\widetilde{\Delta}_{\rho-1});$$

$$\sigma=\sum_{j=0}^{n-1}\operatorname{sign}(\Delta_{j-1}\Delta_j)\quad (\operatorname{sign}0=0)^*.$$

4. Exhaustive descriptions of semidefinite and definite (finite and infinite) Toeplitz matrices have been obtained in the theory of the trigonometric moment problem (see, for example, ([4])) by finding integral representations of their elements. An analogous problem for indefinite Toeplitz forms (matrices) with a fixed number of squares of one sign was solved in papers ([5–8]). In doing so, however, finite forms of the form (2) of rank $\rho(>0)$ for which $\Delta_{\rho-1}=0$ were left aside. The propositions 1^0 and 4^0 given in Sec. 2 make it possible to fill this gap.

From Proposition 4^0, in particular, it follows that for $\rho > 0$ and $\Delta_{\rho-1} = 0$, important characteristics of the matrix $\|c_{p-q}\|_{p,q=0}^{n-1}$, in addition to its order n and rank ρ (it is easy to see that under our assumptions necessarily $n \geq 3$ and $\rho \geq 2$), are the integers $r(\geq 0)$ and $k(> 0)$, determined by the relations

$$\Delta_{r-1} \neq 0; \quad \Delta_r = \Delta_{r+1} = \dots = \Delta_{\rho-2} = \Delta_{\rho-1} = 0; \quad \rho = r + 2k. \tag{5}$$

Theorem 2. *The general form of a Hermitian Toeplitz matrix $\|c_{p-q}\|_{p,q=0}^{n-1}$ of rank $\rho(> 0)$ with minor $\Delta_{\rho-1} = 0$, satisfying conditions (5) with given integers $r(\geq 0)$ and $k(> 0)$, is determined as follows:*

1) *For $r \geq 1$ an arbitrary nonsingular (Hermitian) Toeplitz matrix $\|c_{p-q}\|_{p,q=0}^{r-1}$ is specified (in the case $r = 0$, this step is omitted).*

2) *As the element $c_r(= \bar{c}_{-r})$, for $r \geq 1$, an arbitrary solution ξ of the equation** is chosen*

$$\Delta_r(\xi) \equiv \begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-r+1} & \xi \\ c_1 & c_0 & \cdots & c_{-r+2} & c_{-r+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \xi & c_{r-1} & \cdots & c_1 & c_0 \end{vmatrix} = 0,$$

and for $r = 0$ we put $c_0 = 0$.

3) *The elements $c_{r+1}, c_{r+2}, \dots, c_{n-k-1}$ are determined uniquely by the condition that all matrices $\|c_{p-q}\|_{p,q=0}^{j}$ $(j = r+1, r+2, \dots, n-k+1)$ preserve one and the same rank r***.*

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* In this last formula (for the signature σ) remains valid also in the case when no signs are assigned to minors from among $\{\Delta_j\}$ $(j = 0, 1, \dots, n-1)$ that have become zero; this makes it possible to find the signature σ of a Toeplitz form without knowing its rank ρ.

** It is not difficult to see that this equation, in the case $\Delta_{r-2} \neq 0$, defines a circle, and in the case $\Delta_{r-2} = 0$, a straight line in the ξ-plane (cf. ([6]), p. 471).

*** See ([6]), Theorem 5.8.

4) The element c_{n-k} is chosen so that the rank of the matrix $\|c_{p-q}\|_{p,q=0}^{n-k}$ already exceeds r (while otherwise arbitrary). This rank is always equal to $r+2$, so that for $k = 1$ we have $r+2 = r+2k = \rho$ and $\Delta_{\rho-1} = \Delta_{r+1} = 0$.

5) If $k > 1$, then the remaining elements $c_{n-k+1}, c_{n-k+2}, \dots, c_{n-1}$ may be chosen arbitrarily. In this case the ranks of the matrices $\|c_{p-q}\|_{p,q=0}^{n-k+j}$ turn out to be equal to $r + 2(j+1)$ $(j = 0, 1, 2, \dots, k-1)$, respectively.

Let us explain that, since $\rho - 1 = r + 2k - 1 \geq r + 1$, for $\rho \leq n - k$ the rank of the matrix $\|c_{p-q}\|_{p,q=0}^{\rho-1}$, according to item 3) of Theorem 2, is equal to $r(< \rho)$, while for $\rho = n - k + j$ $(j = 1, 2, \dots, k - 1)$ this rank, according to item 5), is equal to $r + 2j(\leq r + 2k - 2 < \rho)$. Thus, in all cases we have $\Delta_{\rho-1} = 0$.

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Note: Figure translations are in progress. See original paper for figures.