ON THE APPROXIMATION AND CONVERGENCE OF THE GRID METHOD IN ELLIPTIC PROBLEMS

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.949.8

MATHEMATICS

Yu. K. DEM' YANOVICH

ON THE APPROXIMATION AND CONVERGENCE OF THE GRID METHOD IN ELLIPTIC PROBLEMS

(Presented by Academician V. I. Smirnov on 6 XII 1965)

In papers ([1–5, 20]) projection methods for constructing difference equations were considered. In particular, in the case of the Dirichlet problem in a domain Ω with twice continuously differentiable boundary, convergence ([3, 5]) of the grid method in $\mathring{W}_2^1(\Omega)$ with rate $h^{1/2}$ was established. Using R. Courant' s proposed ([1]) linear extension of a grid function over triangles, in the two-dimensional case for the third boundary-value problem L. A. Oganesyan constructed ([20]) equations of the grid method with convergence rate h in $W_2^1(\Omega)$ and condition number of order h^{-3}.

The purpose of the present paper is to obtain, in the n-dimensional case, a theorem on the approximation of functions of the class $W_2^1(\Omega)$ by functions obtained by linear extension over simplices of grid functions, and, in the case of the Dirichlet and Neumann problems, to construct stable ([5]) schemes of the grid method converging with rate h in $W_2^1(\Omega)$, and also to consider some properties of the indicated schemes. Below we shall use the notation of note ([3]).

1°. **Theorem 1.** *For any finite n-dimensional domain Ω with twice continuously differentiable boundary S, there exists a family $\{R_h\}$ of simplicial complexes of the form $R_h = P_h \cup Q_h$, $\mathrm{mes}\,|P_h \cap Q_h| = 0$, where Q_h is an arbitrary simplicial subdivision of the cellular complex whose cells are cubes of a grid with step h, and P_h is a simplicial complex lying in a boundary strip of width c_0h, such that: 1) the body $|R_h|$ of the complex R_h lies in Ω; 2) the set $\Omega \setminus |R_h|$ lies inside a strip of width c_1h^2; 3) the length l of any edge and the value α of any angle between edges with a common endpoint of the complex R_h satisfy the inequalities $c_2h \le l \le c_3h$, $\alpha_0 \le \alpha \le \alpha_1$, where $c_0, c_1, c_2, c_3, \alpha_0$, and α_1 are positive numbers independent of the choice of edges and angles in R_h, and also of the step h, with $0 < \alpha_0 \le \alpha_1 < \pi$.*

We shall say that a function ψv is obtained by linear extension over simplices of a grid function v, given on the zero-dimensional skeleton of the complex R_h, if for each of its simplices T

$$(\psi v)(x) = \sum_{i=0}^{n} \lambda^i v(x_i), \qquad x \in T,$$

where $x_0, \dots, x_n$ are the vertices of the simplex T, and $\lambda^0, \dots, \lambda^n$ are the barycentric coordinates of the point x in this simplex. In what follows $\Omega_{(\varepsilon)}$ denotes the enlargement of the domain Ω to its ε-neighborhood, and $\omega_{W_2^1(\Omega_{(\varepsilon)})}(g,h)$ the modulus of continuity of the function g in the class $W_2^1(\Omega_{(\varepsilon)})$.

Theorem 2. *Let in the domain $\Omega_{(\varepsilon)}$ there be given a family $\{R_h\}$ of simplicial complexes containing the domain Ω, which satisfy condition 3) of the preceding theorem. Let the function g belong to $W_2^1(\Omega_{(\varepsilon)})$. Then for each h $(h < \varepsilon/2c)$ there exists a function defined on the zero-dimensional skeleton of the com-*

of the complex R'_h, a mesh function v, whose linear extension ψv over the simplices of the complex R'_h approximates the function g in such a way that

$$\|g - \psi v\|_{W_2^1(|R_h|)} \leq K \omega_{W_2^1(\Omega_{\varepsilon/2})}(g, ah),$$

where a and K are determined by the numbers c, a_0 and a_1 and do not depend on g and h.

Let now $g \in \overset{\circ}{W}{}_2^1(\Omega)$, and suppose that the boundary S of the domain Ω is three times continuously differentiable. Choose ε so small that the points $x \in \Omega_{(\varepsilon)} \setminus \Omega$ admit a locally unique representation $x = y + \tau m(y)$, $\tau \geq 0$, where $m(y)$ is the exterior normal to the surface S at the point y, while the point $x' = y - \tau m(y)$ locally uniquely determines y and τ. In accordance with the results of V. M. Babich ([6]) and S. M. Nikol' skii ([7]), the function g can be extended to $\Omega_{(\varepsilon)}$ with preservation of the class and of the Hölder exponent by the formula

$$g(y + \tau m(y)) = -g(y - \tau m(y)), \qquad y \in S. \tag{1}$$

Theorem 3. Let the domain Ω have a three-times continuously differentiable boundary. Let the function $g \in \overset{\circ}{W}{}_2^1(\Omega)$ be extended to the domain $\Omega_{(\varepsilon)}$ by formula (1). Suppose, moreover, that a family $\{R_h\}$ of simplicial complexes satisfying the conditions of Theorem 1 is given. Then for each h there exists a mesh function v, defined on the zero-dimensional skeleton of the complex R_h and equal to zero on the boundary of the set $|R_h|$, whose linear extension $\psi_0 v$ over the simplices of R_h, continued by zero to the whole set Ω, approximates the function g in such a way that, for $h < \varepsilon/2c$,

$$\|g - \psi_0 v\|_{W_2^1(\Omega)} \leq K \omega_{W_2^1(\Omega_{\varepsilon/2})}(g, ah) + C_1 h^{1/2} \omega_{W_2^1(\Omega)}(g', a_1 h) +$$

$$+ C_2 \omega_{W_2^1(\Omega)}(g', a_2 h^2),$$

where the constants K, C_1, C_2, a, a_1, and a_2 do not depend on g and h, and by g' is denoted the extension of the function g by zero to the exterior of the domain Ω.

2°. Let us apply Theorems 1-3 to the grid method for the Dirichlet problem

$$A_0 u \equiv -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + au = f, \qquad u|_S = 0, \tag{2}$$

understood in the same way as in ([8]). With respect to the functions a_{ik}, a, and f, the assumptions of the note ([3]) are retained.

Theorem 4. Let the boundary S of the domain Ω be three times continuously differentiable, and let the assumptions of the preceding theorem and of Theorem 3 of the note ([3]) be fulfilled. For the function v_*, solving the problem of minimizing the functional

$$F(\psi_0 v) = (A_0^{1/2}\psi_0 v, A_0^{1/2}\psi_0 v) - 2(f, \psi_0 v)$$

in the space $\bar{X}$ of mesh functions v equal to zero on the boundary of the set $|R_h|$, the estimate

$$\|u^* - \psi_0 v_*\|_{W_2^1(\Omega)} \leq Ch\|f\|_{L_2(\Omega)}$$

is valid, where the positive constant C depends only on the data of the exact problem (2), and u^* denotes the solution of problem (2).

An analogous assertion is valid for the Neumann problem, and also in the case of the Dirichlet problem for a strongly elliptic system of second-order differential equations.

Remark. Under the conditions of Theorem 4 and under the condition that the family $\{R_h\}$ satisfies the assumptions of Theorem 1, the grid method is stable in the sense of the definition formulated in ([5]). In the one-dimensional case, the subspaces given in ([5]) coincide with the subspaces constructed earlier by A. A. Samarskii ([9]).

3°. Let us note that the difference schemes obtained here, as well as in the works ([3,5]), fit into the general theory of difference schemes developed by A. N. Tikhonov and A. A. Samarskii ([9–16]). In what follows we shall use the concepts of this theory ([16]).

Define the operator $\overline{A}_0$ by the equality

$$(\overline{A}_0 v, v_1)_0 = (A_0^{1/2}\psi_0 v, A_0^{1/2}\psi_0 v_1), \qquad v,\ v_1 \in \overline{X}, \tag{3}$$

where the brackets $(...)_0$ denote the scalar product in the space $\overline{X}$, introduced in (5). The operator $\overline{A}_0$ has the form

$$\overline{A}_0 v(x_0) = \sum_{y_0 = \mathfrak{M}(x_0)} A_{x_0 y_0} v(y_0), \tag{4}$$

where $A_{x_0 y_0}$ are functionals on the coefficients of the operator A_0, and $\mathfrak{M}(x_0)$ is a certain set of nodes, called the mesh stencil.

Scheme (4) is called ([16]) homogeneous if $A_{x_0 y_0} = A_{y_0}(\mathfrak{A}_0)$, where $\mathfrak{A}_0$ is the vector of coefficients of the operator A_0, transformed by means of the mapping χ_{x_0}

$$(\chi_{x_0}\varphi)(s) = \varphi(x_0 + sh), \qquad s \in \mathfrak{Z}_0;$$

$\mathfrak{Z}_0$ is a certain set of n-dimensional vectors independent of h, called the reduced coefficient stencil; A_{y_0} is a functional on the space K_0 of vector-functions defined on $\mathfrak{Z}_0$; $\mathfrak{M}(x_0)$ is the set of vectors of the form $y = x_0 + sh,\ s \in \mathfrak{M}_0$; here $\mathfrak{M}_0$ is the reduced mesh stencil.

Theorem 5. *Let the mesh be regular, i.e. let the stars of the complex with vertices inside the polyhedron R_h be congruent to one another. Then the mesh scheme (3) is homogeneous, conservative, and linear, with stencil functionals of rank ∞.*

Let the mapping χ be given on the set of functions continuous in Ω by the formula $(\chi u)(x_0) = u(x_0)$, $u \in C(\Omega)$.

There are known ([9–18]) schemes with a positive definite operator which converge in the class of discontinuous coefficients and have, on smooth functions, first-order approximation. There is an example ([16,17]) of schemes which do not converge in the class of discontinuous coefficients but have second-order approximation. For the schemes considered here the following is established.

Theorem 6. *Let, in some subdomain Ω' of the domain Ω, the mesh be regular. Then for the mesh scheme (3), for $u \in C_4(\Omega')$, the representation holds*

$$(\overline{A}_0 \chi u)(x_0) = (\chi A_0 u)(x_0) + \alpha(h), \tag{5}$$

where

$$\alpha(h) \to 0, \qquad \text{as } h \to 0, \qquad \text{if } a_{ik} \in C_1(\Omega'), \quad a \in C(\Omega'), \tag{6}$$

$$\alpha(h) = O(h), \quad \text{if } a_{ik} \in C_2(\Omega'), \quad a \in C_1(\Omega'), \tag{7}$$

$$\alpha(h) = O(h^2), \quad \text{if } a_{ik} \in C_3(\Omega'), \quad a \in C_2(\Omega'), \tag{8}$$

and the asymptotics indicated here for $\alpha(h)$ are uniform with respect to $x_0 \in \Omega'$.

Remark 1. A statement analogous to Theorem 6 is valid for the schemes given in ([3]).

Remark 2. In the case of families of meshes that are not regular, only the estimates (6), (7) hold.

For a smooth solution u^* of problem (2), the fact of approximation for $n = 1$ implies convergence of the mesh method in the metric $C(\Omega)$, with the order of approximation. For $n \geq 2$ the same assertion is valid in the case where the maximum principle holds.

Theorem 7. *Let the internal normals to the $(n-1)$-dimensional faces of each simplex T of the complex R_h form nonobtuse angles in the sense of the scalar*

product

$$(\xi, \eta)_T = \sum_{i,j=1}^{n} \left(\int_T a_{ij}\, dx \right) \xi_i \eta_j;$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$ are n-dimensional vectors. Then, for the operator $\bar{A}_0$ constructed here, the maximum principle is satisfied.

Remark. For $n = 2$, the fulfillment of the conditions of Kh. L. Smolitskii ((19), p. 98) implies the validity of the conditions of Theorem 7. In the case of a mesh consisting of right triangles with one of the sides parallel to the x_1-axis, from Theorem 7 one easily obtains the following simple sufficient conditions for the maximum principle to hold:

$$3a_{11} \geq a_{22}, \qquad |a_{12}| \leq a_{22}/\sqrt{3}.$$

In conclusion I express my deep gratitude to Prof. S. G. Mikhlin for his attention to this work.

Leningrad State University
named after A. A. Zhdanov

Received
2 XII 1965

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Note: Figure translations are in progress. See original paper for figures.