ON PSEUDODIFFERENTIAL OPERATORS OF PRINCIPAL TYPE

MATHEMATICS

1966

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Abstract

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UDC 517.946.9

MATHEMATICS

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ON PSEUDODIFFERENTIAL OPERATORS OF PRINCIPAL TYPE

(Presented by Academician I. G. Petrovskii on February 9, 1966)

Consider a smooth manifold Ω, on which two complex vector bundles E and F with infinitely differentiable structure are defined. Following L. Hörmander ([1]), we shall call a continuous mapping P from E to F a **pseudodifferential operator** if, from the space of finite sections $C_0^\infty(\Omega, E)$ into $C^\infty(\Omega, F)$, the function $e^{-i\tau\varphi}P(ue^{i\tau\varphi})$ admits an asymptotic expansion in powers of τ as $\tau \to \infty$, provided $u \in C_0^\infty(\Omega, E)$, $\varphi \in C^\infty(\Omega)$, and $\operatorname{grad}\varphi \neq 0$ on $\operatorname{supp} u$. The **symbol** of the operator P is the formal sum $\sum_0^\infty p^j(x,\xi)$, coinciding with the asymptotic series for $e^{-i(x,\xi)}P(ue^{i(x,\xi)})$ as $\xi \to \infty$. In ([1]) it is shown that the principal part of the symbol $p^0(x,\xi)$ is invariantly defined on the cotangent space to Ω.

As is known, the operator P is called **elliptic** if $p^0(x,\xi)$ effects an invertible mapping: $E_x \to F_x$, when $x \in \Omega$, and $\xi \neq 0$ is a vector cotangent to Ω at the point x. It is known that an operator P of order m is elliptic if and only if

$$\|u\|_{s+m} \leq C(\|Pu\|_s + \|u\|_t), \qquad u \in C_0^\infty(K, E),$$

where K is an arbitrary compact set in Ω; s and t are arbitrary real numbers:

$$\|u\|_s = \left(\int |\tilde{u}(\xi)|^2(1+|\xi|^2)^s\,d\xi\right)^{1/2}.$$

Here

$$\tilde{u}(\xi) = \int u(x)e^{-i(x,\xi)}\,dx$$

is the Fourier transform of the function $u(x)$. In ([2]) L. Hörmander showed that the existence of the estimate

$$\|u\|_s \leq C(\|Pu\|_{s-m+\delta} + \|u\|_t), \qquad u \in C_0^\infty(K, E) \tag{1}$$

for $0 < \delta < 1/2$ is also equivalent to ellipticity of the operator P, but for $\delta = 1/2$ a new class of operators arises, called **subelliptic** by Hörmander. In the present work we describe pseudodifferential operators for which estimate (1) holds with $0 \le \delta < 1$. These operators should naturally be called **operators of principal type**, since for them (and only for them) the existence of an estimate of this form is determined solely by the principal part $p^0(x,\xi)$ of the symbol of the operator P.

Theorem 1. *The estimate (1) with $\delta = k/(k+1)$ holds if and only if, for every compact subset $K \subset \Omega$, there exists a constant C such that, for $x \in K$ and $0 \ne \xi \in R^n$,*

$$\int |\psi(y)|^2\,dy \le C \int \left| \sum_{|\alpha|+|\beta|\le k} \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} q(x,\xi)}{\partial\xi^\alpha \partial x^\beta} |\xi|^{k(|\alpha|-|\beta|)/(k+1)} y^\beta D^\alpha \psi(y) \right|^2 dy +$$

$$+\varepsilon(\xi) \sum_{|\alpha|+|\beta|\le k+2} |\xi|^{-2|\alpha|(k-1)/(k+1)} \int |y^\beta D^\alpha \psi(y)|^2\,dy, \qquad \psi \in C_0^\infty(R^n)^e. \quad (2)$$

Here

$$D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right);$$

e is the dimension of the bundle E; $q(x,\xi) = p^0(x,\xi)|\xi|^{-m+k/(k+1)}$; m is the order of the operator P; $\varepsilon(\xi) \to 0$ as $\xi \to \infty$.

Apparently, the operators described by Theorem 1 exhaust all operators of principal type.

Theorem 2. If, for the operator P, estimate (1) holds with $0 < \delta < 2/3$, and $e = f = 1$, then this estimate also holds for $\delta = 1/2$.

In applications the following variant of Theorem 1 is convenient, in which it is not required that the operator have order $k/(k+1)$.

Theorem 1′. For an operator P of order m, estimate (1) holds if and only if, for every compact subset $K \subset \Omega$, there exist a constant C and a function $\varepsilon(\xi)$, tending to zero as $\xi \to \infty$, such that

$$\int |\psi(y)|^2\,dy \le C|\xi|^{-2m+2k/(k+1)} \int \left| \sum_{|\alpha|+|\beta|\le k} \frac{\partial^{\alpha+\beta} p^0(x,\xi)}{\partial\xi^\alpha \partial x^\beta} \frac{1}{\alpha!\beta!} \times \right.$$

$$\left. \times |\xi|^{k(|\alpha|-|\beta|)/(k+1)} y^\beta D^\alpha \psi(y) \right|^2 dy +$$

$$+\varepsilon(\xi)\int\sum_{|\alpha|+|\beta|\leq k+2}|\xi|^{-2|\alpha|(k-1)/(k+1)}\left|y^{\beta}D^{\alpha}\psi(y)\right|^{2}dy,\qquad\psi\in C_{0}^{\infty}(R^{n}).\quad(2')$$

Of course, conditions (2) and (2′) are not always easy to verify. However, they make it possible to reduce the difficult question of estimates of type (1) for a pseudodifferential operator (including for a differential operator of arbitrary order) to the question of the existence of the simpler estimate (2) (or (2′)) for a linear differential operator of order k.

In conclusion we give an example of an application of the results obtained to the study of noncoercive boundary value problems for elliptic equations.

Let Ω be a compact n-dimensional Riemannian manifold with smooth boundary Γ, and let ν be a nondegenerate vector field on Γ. We consider the problem of finding a function $u(x)$ for which

$$\Delta u=f\quad\text{in }\Omega;\qquad\frac{\partial u}{\partial\nu}=g\quad\text{on }\Gamma.$$

Let Γ_0 be a smooth manifold in Γ of dimension $n-2$, and suppose that the field ν is not tangent to Γ at points of $\Gamma\backslash\Gamma_0$, while on Γ_0 it has contact with Γ of order k. This means that in a local coordinate system whose center coincides with some point on Γ_0, and such that the domain Ω is described by the inequality $x_n>0$, while Γ_0 is given by the equations $x_n=x_{n-1}=0$, the boundary condition is written in the form

$$x_{n-1}^{k}\frac{\partial u}{\partial x_n}+\sum_{j=1}^{n-1}\nu_j(x)\frac{\partial u}{\partial x_j}=g(x).$$

Such problems were studied in [3], but there the order of contact was not taken into account. For $k=1$ this problem was considered in [2]. If x_n coincides with the length of the arc of the geodesic normal to Γ, then the function $v(x)=u(x_1,\dots,x_{n-1},0)$ satisfies the equation

$$Pv=g+Lf,$$

where the principal part of the symbol of the operator P is equal to

$$p^{0}(x,\xi)=x_1^{k}|\xi|+i\sum_{j=1}^{n-1}\nu_j(x)\xi_j,\qquad(3)$$

and L is a linear operator $H^{s}(\Omega)\to H^{s+1/2}(\Gamma)$. We require that the field ν not be tangent to Γ_0, i.e., that $\nu_{n-1}(x)\neq0$. It is easy to verify that for the symbol

(3) the inequality (2′) holds if k is odd and $\nu_{n-1}(x) < 0$ on Γ_0, or if k is even and $\nu_{n-1}(x) \neq 0$. Hence it follows that, under the indicated conditions, the solution $u(x)$ of our problem belongs to the space $H^s(\Omega)$, provided only that $f \in H^{s-(k+2)/(k+1)}(\Omega)$ and $g \in H^{s-3/2+k/(k+1)}(\Gamma)$. The example,

given in [3], shows the impossibility of improving this result.

Remark 1. It can be shown that operators of principal type are hypoelliptic (see [4]).

Remark 2. Suppose decompositions of the bundles E and F into direct sums are given,

$$E = \bigoplus_{k=1}^{K} E_k, \qquad F = \bigoplus_{j=1}^{J} F_j$$

and two sequences of real numbers $s = (s_1, \dots, s_K)$, $t = (t_1, \dots, t_J)$. We agree to say that a pseudodifferential operator P from E to F has type (s, t) if, for every compact set K,

$$\|Pu\|_{(t)} \leq C\|u\|_{(s)}, \qquad u \in C_0^\infty(K, E),$$

where

$$\|u\|_{(s)} = \left(\sum_{k=1}^{K} \|u_k\|_{s_k}^2 \right)^{1/2}.$$

The results of the present note are readily extended to such operators if estimate (1) is replaced by the inequality

$$\|u\|_{(s)} \leq C \left(\|Pu\|_{(t+\delta)} + \|u\|_{(r)} \right), \qquad u \in C_0^\infty(K, E),$$

where $t + \delta = t_1 + \delta_1, \dots, t_J + \delta$ and $0 \leq \delta < 1$, and the principal part $p^0(x, \xi) = (p_{jk}^0)_{\substack{k=1,\dots,K \\ j=1,\dots,J}}$ of the symbol $p(x, \xi) = (p_{jk})_{\substack{k=1,\dots,K \\ j=1,\dots,J}}$ is taken to be those terms of the matrix $p(x, \xi)$ for which the order p_{jk}^0 is equal to $s_k - t_j$.

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Received
9 II 1966

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4. Yu. V. Egorov, *DAN*, 168, No. 6 (1966).

Note: Figure translations are in progress. See original paper for figures.