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Abstract

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MATHEMATICS

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PERIODIC NILPOTENT LINEAR GROUPS OVER THE FIELD OF RATIONAL NUMBERS

(Presented by Academician A. I. Mal' tsev on V 22, 1965)

The theory of linear nilpotent and locally nilpotent groups was developed by D. A. Suprunenko ([1–6]). He also studied periodic linear locally nilpotent groups over an algebraically closed field of characteristic zero ([7]).

In the present note we study periodic nilpotent subgroups of the full linear group $GL(n,R)$ over the field of rational numbers R^*. It is proved that two maximal periodic nilpotent subgroups of $GL(n,R)$ are conjugate in $GL(n,R)$ if their centers are conjugate in $GL(n,R)$. It is also shown that in $GL(n,R)$ there exists, up to conjugacy, only a finite number of maximal nilpotent periodic subgroups. A complete description is given of the maximal periodic nilpotent subgroups of $GL(n,R)$.

1. We first consider irreducible nilpotent periodic subgroups of $GL(n,R)$. Their structure is closely connected with the structure of linear p-groups over the field of rational numbers (see Theorem 1). (For results on linear p-groups see ([9]), where Sylow p-subgroups of the full linear group over an arbitrary field are studied.)

The structure of irreducible nilpotent periodic subgroups is described by the following theorem.

Theorem 1. Let Γ be an irreducible periodic nilpotent subgroup of $GL(n,R)$. Then Γ is conjugate in $GL(n,R)$ to a certain subgroup of the group of all matrices g of the form

$$g = a_1 \times a_2 \times \cdots \times a_\nu, \tag{1}$$

where $\times$ is the sign of the Kronecker product, a_i ranges over an irreducible p_i-subgroup of a Sylow subgroup of the group $GL(\varphi(p_i^{\alpha_i}),R)$ (φ is Euler' s function). Consequently,

$$n = \varphi(k), \tag{2}$$

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where $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\nu^{\alpha_\nu}$ is the canonical decomposition of the number k (see (1)), and the group of all matrices of the form (1) is determined uniquely, up to conjugacy in $GL(n, R)$, by specifying the number k from (2).

Corollary. If $n \neq \varphi(k)$, where k is an integer and φ is Euler' s function, then in $GL(n, R)$ there are no irreducible nilpotent periodic subgroups.

Now the following question naturally arises: which groups of matrices of the form (1) are maximal irreducible nilpotent periodic subgroups of $GL(n, R)$, or, in other words, to which numbers k satisfying the condition $n = \varphi(k)$ does there correspond a maximal irreducible nilpotent periodic subgroup of $GL(n, R)$.

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* We note that every locally nilpotent subgroup of $GL(n, R)$ is nilpotent ([6,8]).

Let us turn to the solution of this question. It is easy to see that the group of matrices of the form (1) can be regarded as a subgroup of $GL(r, \Sigma)$, where

$$\Sigma = R(\varepsilon), \tag{3}$$

ε is a primitive root of unity of degree m, and (cf. (2))

$$m = p_1 p_2 \cdots p_\nu, \tag{4}$$

$$r = n / \varphi(m). \tag{5}$$

We shall first show that the group of all matrices of the form (1) is almost always maximal among irreducible nilpotent periodic subgroups whose linear R-envelope of the center coincides with Σ. More precisely, the following is true.

Lemma 1. *Let Γ be the group of all matrices of the form (1), determined by such a number k that $n = \varphi(k)$, and let $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$ be the canonical factorization of the number k, and let $\Sigma = R(\varepsilon)$, where ε is a primitive root of unity of degree $m = p_1 p_2 \cdots p_\nu$. Then Γ is maximal among irreducible nilpotent periodic subgroups of the group $GL(n, R)$ whose linear R-envelopes of the centers coincide with Σ, except for the case when Γ is determined by a number k satisfying the relation $n = \varphi(k) = \varphi(2k)$.*

Now one can answer the question formulated earlier. The answer is given by the following

Theorem 2. *For any k such that $n = \varphi(k)$, the group of all matrices of the form (1), determined by this number k, is a maximal irreducible nilpotent periodic subgroup of $GL(n, R)$, except for the case when the group (1) is determined by such a number k for which the relation $n = \varphi(k) = \varphi(2k)$ is possible.*

From Theorems 1 and 2 there follows the following

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Theorem 3. *Two maximal irreducible nilpotent periodic subgroups of $GL(n,R)$ are conjugate in $GL(n,R)$ if their centers are conjugate in $GL(n,R)$.*

The theorems formulated above give a complete classification and construction of maximal irreducible nilpotent periodic subgroups of $GL(n,R)$. For convenience, let us combine the results obtained as follows.

First main theorem. *If $n \neq \varphi(k)$, where φ is Euler' s function and k is an integer, then in $GL(n,R)$ there are no irreducible nilpotent periodic subgroups. Let $\mathfrak{M}$ be the set of all such numbers k that $n = \varphi(k)$, except for that number k_1 for which the relation $n = \varphi(k_1) = \varphi(2k_1)$ is possible, and let*

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}, \tag{6}$$

be the canonical factorization of the number k.

Then:

1. *For any k from $\mathfrak{M}$, in $GL(n,R)$ there is a maximal irreducible nilpotent periodic subgroup whose matrices can be represented in the form*

$$g = a_1 \times a_2 \times \cdots \times a_\nu,$$

where a_i runs through a p_i-subgroup of the Sylow group $GL(\varphi(p_i^{\alpha_i}), R)$ (see (6)), and $\times$ denotes the Kronecker product.

2. *Different k from $\mathfrak{M}$ determine nonisomorphic groups.*
3. *Every maximal irreducible nilpotent periodic subgroup of $GL(n,R)$ is conjugate in $GL(n,R)$ to the group determined by some k from $\mathfrak{M}$.*

Corollary. In $GL(n,R)$, up to conjugacy, there exists only a finite number of maximal irreducible nilpotent periodic subgroups.

II. Let now Γ be a maximal nilpotent periodic subgroup of $GL(n,R)$. As is known, any periodic subgroup of $GL(n,R)$ is completely reducible ([13], [14]). Consequently, Γ is a completely reducible group.

Let G be one of the irreducible components of the group Γ. If m is the degree of G, then $m = \varphi(k)$, and G is a maximal irreducible nilpotent periodic subgroup of $GL(m,R)$. Consequently, a maximal periodic nilpotent subgroup of the group $GL(n,R)$ is uniquely determined, up to conjugacy in $GL(n,R)$, by the degrees $n_1, n_2, \dots, n_s$ of its irreducible components and by the centers of these irreducible groups, or, in other words, by a set of numbers $k_1, k_2, \dots, k_s$ such that, for every k_i,

$$\varphi(k_i) \neq \varphi(2k_i), \tag{7}$$

where the numbers n_i, where

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$$n_i = \varphi(k_i), \tag{8}$$

satisfy the relation

$$n = n_1 + n_2 + \dots + n_s, \qquad n_i > 0. \tag{9}$$

Thus, it is necessary to determine to what set of numbers $k_1, k_2, \dots, k_s$, satisfying (7), (8), and (9), there corresponds a maximal nilpotent periodic subgroup of $GL(n, R)$. We shall formulate this assertion more precisely.

Let n be represented in the form (9), with each k_i from (8) satisfying (7). Represent the space R^n, in which $GL(n, R)$ acts, in the form

$$R^n = \Sigma_1 + \dots + \Sigma_s, \tag{10}$$

where Σ_i is a subspace of dimension n_i. By Γ_{k_i} we shall denote the subgroup of $GL(n, R)$ which induces on each Σ_j $(j \neq i)$ the identity group, and on Σ_i the maximal irreducible nilpotent periodic group of degree n_i determined by the number k_i, with $n_i = \varphi(k_i)$.

Construct the group

$$\Gamma_{k_1,k_2,\dots,k_s} = \Gamma_{k_1}\Gamma_{k_2}\cdots\Gamma_{k_s}. \tag{11}$$

It turns out that the group $\Gamma_{k_1,k_2,\dots,k_s}$ is not always a maximal nilpotent periodic subgroup of $GL(n, R)$. Our task is to find conditions under which the sets of numbers $k_1, k_2, \dots, k_s$, satisfying (7), (8), and (9), determine a maximal nilpotent periodic subgroup $\Gamma_{k_1,k_2,\dots,k_s}$ of the group $GL(n, R)$. The method used by us for this goes back to ([15]).

Lemma 2. If the formula

$$i \neq j \Rightarrow k_i \neq k_j$$

is true, then $\Gamma_{k_1,k_2,\dots,k_s}$ is a maximal nilpotent periodic subgroup of $GL(n, R)$.

Lemma 3. Let $k_1 = k_2 = \dots = k_s = k$. The group $\Gamma_{k_1,k_2,\dots,k_s}$ is contained in some nilpotent periodic subgroup of the group $GL(n, R)$ only when $k = 2^\alpha,\ s = 2^\beta$.

From the last two lemmas it follows:

Second main theorem. A reducible nilpotent periodic group $\Gamma_{k_1,\dots,k_s}$, inducing in each irreducible block of degree $n_i > 0$ a maximal irreducible nilpotent periodic group determined by the number k_i, where $n_i = \varphi(k_i)$, is not only then ...

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is a maximal nilpotent periodic subgroup of $GL(n, R)$, $n = n_1 + \dots + n_s$, when among $k_1, k_2, \dots, k_s$ there are at least two such numbers k_{j_1} and k_{j_2} that $k_{j_1} = k_{j_2} = 2^\alpha$.

At the same time, the following assertions follow from this.

Theorem 4. In $GL(n, R)$, up to conjugacy in $GL(n, R)$, there exists only a finite number of maximal nilpotent periodic subgroups.

Theorem 5. Two maximal nilpotent periodic subgroups of $GL(n, R)$ are conjugate in $GL(n, R)$ if their centers are conjugate in $GL(n, R)$.

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