DETERMINATION OF A DIRAC SYSTEM FROM THE SCATTERING PHASE

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DETERMINATION OF A DIRAC SYSTEM FROM THE SCATTERING PHASE

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In the present paper the inverse problem of scattering theory is solved for the system of Dirac equations. This problem was studied earlier in papers ($^{1-3}$), in which, however, no definitive results were obtained. For the Sturm-Liouville equation a similar problem was solved by V. A. Marchenko (^{4}).

§ 1. In paper (^{5}) the canonical form of a system of Dirac equations was indicated. Therefore from the very beginning we shall assume that the Dirac system has been reduced to the canonical form:

$$B\frac{d}{dx}y + mTy + Qy = \lambda y, \qquad 0 \le x < \infty, \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

and m is a constant number (mass). Let the estimates

$$|p(x)| \le C/(1+x)^{2+\varepsilon}, \qquad |q(x)| \le C/(1+x)^{1+\varepsilon}, \tag{2}$$

hold for the real functions p and q, where C and ε are positive constants. Consider the boundary-value problem generated by equation (1) and the boundary condition

$$y_1(0) = 0. \tag{3}$$

It is known that under conditions (2) the problem (1)–(3) has a continuous spectrum covering $(-\infty, -m)$ and (m, ∞), and a finite number $\lambda_1, \dots, \lambda_n$ of simple eigenvalues situated in the interval $(-m, m)$. Denote by

$$\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$$

the solution of equation (1) with the initial conditions $\varphi_1(0) = 0$, $\varphi_2(0) = 1$. This solution satisfies the boundary condition (3), and therefore $\varphi(x,\lambda_k)$ ($k = 1,\dots,n$) is an eigenfunction of the problem (1)–(3).

Denote the norm of this function by M_k:

$$\mu_k = \int_0^\infty \{\varphi_1^2(x,\lambda_k) + \varphi_2^2(x,\lambda_k)\}\,dx.$$

In the case when λ belongs to the continuous spectrum, i.e. to the intervals $(-\infty,-m)$ and (m,∞), the function $\varphi(x,\lambda)$ as $x \to \infty$ has the asymptotic form

$$\varphi(x,\lambda) = A(\lambda) \begin{pmatrix} -\sqrt{(\lambda+m)/(\lambda-m)}\sin(kx+\delta) \\ \cos(kx+\delta) \end{pmatrix},$$

where $k = \sqrt{\lambda^2 - m^2}$, and $A(\lambda)$ and $\delta(\lambda)$ are continuous functions for $m < |\lambda| < \infty$. The function $A(\lambda)$ is called the **scattering amplitude**, and $\delta(\lambda)$ the **phase–**

of the scattering problem (1)–(3). In what follows, the collection

$$\delta(\lambda);\ \lambda_1,\dots,\lambda_n;\ \mu_1,\dots,\mu_n$$

will be called the scattering data of problem (1)–(3). In the present paper we indicate necessary and sufficient conditions for the function $\delta(\lambda)$ and the numbers $\lambda_1,\dots,\lambda_n$; $\mu_1,\dots,\mu_n$ to be the scattering data of a problem of type (1)–(3) with a matrix $Q(x)$ satisfying conditions (2).

§ 2. In solving the problem posed above, a fundamental role is played by the transformation operator with a condition at infinity. It is not difficult to verify that equation (1) has a solution $F(x,\lambda) = \binom{F_1}{F_2}$, which as $x \to \infty$ tends to

$$f(x,\lambda) = \begin{pmatrix} -i(\lambda+m)/k \\ 1 \end{pmatrix} e^{-ikx}, \qquad \text{where } k = \sqrt{\lambda^2 - m^2}.$$

It turns out that there exists a matrix function $A(x,t)$ such that

$$F(x,\lambda) = f(x,\lambda) + \int_x^\infty A(x,t) f(t,\lambda)\,dt, \tag{4}$$

and, moreover, for the elements of the matrix $A(x,t)$ the estimates

$$|A_{ij}| \leq C_1/(1+x)(1+t)^{1+\varepsilon}, \qquad i \neq j; \tag{5}$$

$$|A_{ii}| \leq C_1/(1+t)^{1+\varepsilon} \tag{6}$$

hold (C_1 is a constant);

$$BA(x,x) - A(x,x)B = Q(x). \tag{7}$$

For real values of λ and for $|\lambda| > m$, the vector-function $\overline{F(x,\lambda)}$ is a solution of equation (1), linearly independent of $F(x,\lambda)$. Therefore $\varphi(x,\lambda)$ is their linear combination. Using this fact, it is not difficult to prove that $\delta(\lambda) = \arg F_1(0,\lambda)$, and from this it follows at once that each element of the matrix function

$$F_\delta(x) = \int_{-\infty}^{\infty} \left\{ \begin{pmatrix} -\sqrt{(\lambda+m)/(\lambda-m)}\,\sin(kx+\delta) \\ \cos(kx+\delta) \end{pmatrix} \left(-\sqrt{\frac{\lambda+m}{\lambda-m}}\,\sin\delta,\ \cos\delta \right) - \begin{pmatrix} -\sqrt{(\lambda+m)/(\lambda-m)}\,\sin kx \\ \cos kx \end{pmatrix} \right. \tag{8}$$

belongs to $L_2(-\infty,\infty)$. Here

$$\rho'_m(\lambda) = \frac{1}{\pi}\frac{\lambda-m}{k}, \quad |\lambda| \geq m; \qquad \rho'_m(\lambda) = 0, \quad |\lambda| < m.$$

This property of the function $\delta(\lambda)$ and Parseval' s equality

$$\int_{-\infty}^{\infty} \frac{\varphi(x,\lambda)\tilde{\varphi}(y,\lambda)}{A^2(\lambda)}\,d\rho_m(\lambda) + \sum_{k=1}^{n} \mu_k \varphi(x,\lambda_k)\tilde{\varphi}(y,\lambda_k) = I\delta(x-y) \tag{9}$$

allow us to derive the integral equation for the kernel

$$F(x+y) + A(x,y) + \int_x^{\infty} A(x,t)F(t+y)\,dt = 0, \tag{10}$$

where

$$F(x+y) = F_\delta(x+y) + \sum_{k=1}^{m} \mu_k \begin{pmatrix} \sqrt{(\lambda_k+m)/(m-\lambda_k)} \\ 1 \end{pmatrix} \left(\sqrt{(\lambda_k+m)/(m-\lambda_k)},\ 1 \right) e^{-\sqrt{m^2-\lambda_k^2}(x+y)}. \tag{11}$$

This integral equation gives us the possibility of formally solving the inverse problem from the scattering data. Indeed, equation (10) is constructed only from the scattering data, and if it has a unique solution $A(x,t)$, then the vector-function $F(x,\lambda)$, defined by formula (4), satisfies an equation of type (1) with a matrix determined by formula (7). However, in this connection the question remains open whether in fact $\delta(\lambda); \lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$ are scattering data for an equation of the form (1). To answer this question, it is necessary to study the properties of the scattering data in greater detail.

Theorem 1. The scattering data $\delta(\lambda), \lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$ of problem (1)–(2) have the following properties:

1. Each element of the symmetric matrix function $F_\delta(t)$, defined by formula (8), belongs to $L_2(-\infty,\infty)$, and for $x \geq 0$ the following estimates hold for the elements of $F_\delta(x)$:

$$|F_{ii}(x)| \leq C_2/(1+x)^{1+\varepsilon}, \qquad i = 1, 2; \tag{12}$$

$$|F_{12}(x)| = |F_{21}(x)| \leq C_3/(1+x)^{2+\varepsilon}, \tag{13}$$

where C_2 and C_3 are constants.

2. For all $x \geq 0$ the homogeneous equation

$$(f_1(y),\, f_2(y)) + \int_x^\infty (f_1(t),\, f_2(t)) F(t+y)\, dt = 0 \tag{14}$$

has only the zero solution in $L_2(x,\infty)$.

3. The number of linearly independent solutions from $L_2(0,\infty)$ of the homogeneous equation

$$(f_1(y),\, f_2(y)) + \int_0^\infty (f_1(t),\, f_2(t)) F_\delta(t+y)\, dt = 0 \tag{15}$$

is equal to the number n of eigenvalues of problem (1)–(3).

4. The homogeneous equation

$$-(f_1(y),\, f_2(y)) + \int_{-\infty}^0 (f_1(t),\, f_2(t)) F_\delta(t+y)\, dt = 0 \qquad (-\infty < y \leq 0) \tag{16}$$

has only the zero solution in $L_2(-\infty,0)$.

We outline the proof of this theorem. The first part of property 1 is obvious and has already been formulated by us earlier. To obtain the estimates (12) and (13), it is necessary, in the main equation (11), to put $y = x$ and, regarding

$F(x)$ as unknown, solve the resulting equation by the method of successive approximations. Then the estimates (12)–(13) follow directly from the estimates (5)–(6) for $A(x,t)$.

We prove property 2. Suppose the homogeneous equation (14) has a nonzero solution $(f_1(t), f_2(t))$ from $L_2(x,\infty)$. Multiply this equation by $(f_1(y), f_2(y))$ scalarly and integrate over (x,∞). Then we obtain:

$$\int_{-\infty}^{\infty}\{\Phi(\lambda)\cos\delta + G(\lambda)\sin\delta\}^2 d\rho_m(\lambda) + \sum_{k=1}^{m}\mu_k a_k^2 = 0, \tag{17}$$

where

$$a_k = \int_x^{\infty}\{f_1(t)f_1(t,\lambda_k) + f_2(t)f_2(t,\lambda_k)\}\,dt; \tag{18}$$

$$\Phi(\lambda) = \int_x^{\infty}\left\{-f_1(t)\sqrt{\frac{\lambda+m}{\lambda-m}}\sin kt + f_2(t)\cos kt\right\}dt; \tag{19}$$

$$G(\lambda) = \int_x^{\infty}\left\{-f_1(t)\sqrt{\frac{\lambda+m}{\lambda-m}}\cos kt - f_2(t)\sin kt\right\}dt. \tag{20}$$

It follows from (17) that all $a_k = 0$ and $\Phi(\lambda)\cos\delta + G(\lambda)\sin\delta = 0$. Obviously,

$$\Phi(\lambda)\cos\delta + G(\lambda)\sin\delta = \mathrm{Re}\{(\Phi + iG)e^{-i\delta}\}$$

$$= \frac{1}{2}\left[(\Phi + iG)e^{-i\delta} + (\Phi - iG)e^{i\delta}\right] \equiv 0.$$

Hence

$$(\Phi + iG)/(\Phi - iG) = -e^{2i\delta} = -F_1(0,\lambda)/\overline{F_1(0,\lambda)}. \tag{21}$$

Further, it is clear that $\Phi(\lambda_k) + iG(\lambda_k) = a_k = 0$. This means that the zeros of $F_1(0,\lambda)$ are zeros of the function $\Phi + iG$. Therefore

$$iP(\lambda) = [\Phi(\lambda) + iG(\lambda)]/F_1(0,\lambda)$$

is an analytic function in the upper half-plane, continuous up to the real axis. For real λ, it is evident from (21) that $P(\lambda) = \overline{P(\lambda)}$. Therefore $P(\lambda)$ is an entire function. On the other hand, $P(\lambda) \to 0$ as $|\lambda| \to \infty$. Hence $P(\lambda) \equiv 0$.

Consequently, $\Phi(\lambda)+iG(\lambda) \equiv 0$, i.e. $\Phi(\lambda) \equiv 0$ and $G(\lambda) \equiv 0$. But then $f_1(t) \equiv 0$, $f_2(t) \equiv 0$. A contradiction is obtained. Property 2 is proved.

Properties 3 and 4 are proved analogously.

It can be proved that the properties of the scattering data listed in Theorem 1 are not only necessary, but also sufficient for $\delta(\lambda); \lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$ to be scattering data of a problem of type (1)–(3) with a matrix $Q(x)$ satisfying condition (2). We shall not dwell on this here.

§ 3. In conclusion, we note that we have also succeeded in solving the inverse problem by scattering data for the Dirac system in the case when its coefficients have singularities of the form

$$\begin{pmatrix} 0 & -l/x \\ -l/x & 0 \end{pmatrix}$$

at zero and at infinity. It turned out that the scattering data of the problem without a singularity are the scattering data of the problem with a singularity of the indicated type, and conversely.

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Note: Figure translations are in progress. See original paper for figures.

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