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ON SELF-SIMILAR MOTION OF A RELATIVISTIC GAS IN A COMOVING COORDINATE SYSTEM

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For the problem of stellar collapse at a late stage of their evolution ([1,2]), it is of interest to consider centrally symmetric self-similar gas motions in a comoving frame of reference with interval

$$ds^2 = c^2 d\tau^2 e^{\sigma} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - e^{\omega} dR^2. \tag{1}$$

In this system, as is known, there is no singularity of the metric of the Schwarzschild type, which makes it possible to trace the motion of the gas up to infinite densities at the center. The indicated frame of reference was used, in particular, by V. A. Skripkin in solving the problem of a point explosion in an ideal incompressible fluid in the general theory of relativity ([3]). The system of Einstein equations in comoving coordinates, in contrast to the equations of motion, is rather complicated ([4]), which makes its self-similar investigation extremely difficult.

Greater possibilities, in our view, are opened up by the transformation of the indicated system proposed by M. A. Podurets ([5]). In ([5]) the mass of gas m enclosed within a volume with current radius R is introduced into consideration,

$$m = \frac{1}{2}\frac{c^2 r}{G}\left[1 + e^{-\sigma}\frac{\dot{r}^2}{c^2} - e^{-\omega}{r'}^2\right]. \tag{2}$$

Here c is the speed of light, G is the Newtonian gravitational constant, and a dot and a prime denote derivatives with respect to τ and R, respectively.

With the aid of (2), the cumbersome system of equations ([4]) assumes an exceptionally simple and physically transparent form (see ([5]))

$$\dot{m} = \frac{4\pi}{c^2} p r^2 \dot{r}, \quad m' = -\frac{4\pi}{c^2}\varepsilon r^2 r', \quad \sigma' = -2\frac{p'}{p+\varepsilon}, \quad \dot{\omega} = -2\frac{\dot{\varepsilon}}{p+\varepsilon} - 4\frac{\dot{r}}{r}, \tag{3}$$

where p is the pressure and ε the energy density. The system (2), (3) was used by M. A. Podurets for calculation by computer of the collapse of a cold Fermi gas ([6]).

In the present note the system of equations (2), (3) is investigated by constructing a self-similar solution, which in some cases makes it possible to find an exact solution and to analyze comparatively simply the motion of the gas in its own gravitational field.

According to the general theory of dimension and similarity ([7a]), in the system (2), (3) the constants determining the motion are

$$[a] = [{}^{1}/_{2}c^2/G] = [M/L], \quad [c] = [L/T]. \tag{4}$$

The first of them contains the symbol of mass; therefore it is used for constructing self-similar expressions for m, p, ε; with the aid of the second constant c, the only possible dimensionless parameter is constructed,

$$\lambda = R/c\tau. \tag{5}$$

For the unknown quantities we have

$$\begin{aligned} m = aRM(\lambda), \qquad p = a\tau^{-2}P(\lambda), \qquad \varepsilon = a\tau^{-2}E(\lambda),\\ r = R\rho(\lambda), \qquad e^{-\sigma} = X(\lambda), \qquad e^{-\omega} = Y(\lambda). \end{aligned} \tag{6}$$

Here M, P, E, ρ, X, Y are dimensionless functions of the parameter λ.

Substituting (6) into (2), (3), we arrive at the system of ordinary equations

$$M = \rho\left[1 + X\lambda^4\rho_\lambda^2 - Y(\rho + \rho_\lambda\lambda)^2\right], \tag{7}$$

$$M_\lambda = -4\pi\lambda^2 P\rho^2\rho_\lambda, \tag{8}$$

$$M + \lambda M_\lambda = 4\pi\lambda^2 E\rho^2(\rho + \rho_\lambda\lambda), \tag{9}$$

$$X_\lambda X^{-1} = 2P_\lambda(P + E)^{-1}, \tag{10}$$

$$Y_\lambda Y^{-1} = 4E(E + P)^{-1}\lambda^{-1} + 2E_\lambda(E + P)^{-1} + 4\rho_\lambda\rho^{-1}, \tag{11}$$

where the subscript λ denotes differentiation with respect to λ.

1. As a first example, let us consider the motion of a continuous medium with pressure equal to zero. For this problem an exact solution is known, found by Tolman (4), which makes it possible to compare the self-similar and the exact solutions.

For $p = 0$, it follows from (3) that $\sigma' = 0$, and one may put $Ee^{\sigma} = 1$. Further, from (8) $M_{\lambda} = 0$ and $M = C_1$, i.e.

$$m = aRM = aC_1R. \tag{12}$$

The latter means that for $P = 0$ the mass inside the current radius R is conserved in the course of the motion.

Further, from (9) and (11), with the aid of (12), it follows that

$$E = \frac{M}{4\pi\lambda^2(\rho + \lambda\rho_{\lambda})} = \frac{C_1}{4\pi\lambda^2\rho^2(\rho + \lambda\rho_{\lambda})},$$

$$Y = C_2\lambda^4\rho^4E^2 = B(\rho + \lambda\rho_{\lambda})^{-2}, \qquad B = C_1C_2(4\pi)^{-2}. \tag{13}$$

Now all the unknown quantities are expressed in terms of $r(\lambda)$, for whose determination from (7) we have

$$C_1\rho^{-1} = \left[(1 - B) + \lambda^4\rho_{\lambda}^2\right]. \tag{14}$$

The solution of the last equation has the form

$$\frac{c(\tau_0 - \tau)}{R} = \frac{\sqrt{(B-1)\rho^2 + C_1\rho}}{(B-1)} - \frac{C_1}{(B-1)^{3/2}} \operatorname{Ar\,sh} \sqrt{\frac{(B-1)\rho}{C_1}} \quad \text{for } B - 1 > 0, \tag{15}$$

$$\frac{c(\tau_0 - \tau)}{R} = \frac{\sqrt{(B-1)\rho^2 + C_1\rho}}{(B-1)} + \frac{C_1}{(1-B)^{3/2}} \arcsin \sqrt{\frac{(1-B)\rho}{C_1}} \quad \text{for } B - 1 < 0,$$

$$\rho = \left(\frac{3\sqrt{C_1}}{2}\right)^{2/3} \left[\frac{c(\tau_0 - \tau)}{R}\right]^{-2/3} \quad \text{for } B = 1.$$

If in (15) one replaces $(B-1)$, C_1 by arbitrary functions $f(R)$, $F(R)$, then the solution will coincide with Tolman' s solution. Note that in obtaining (15) τ is replaced by $\tau_0 - \tau$, which is permissible in self-similar solutions. As $\tau \to \tau_0$, for the energy density ε, r, e^{ω}, we naturally obtain the same results as for $\tau_0 = \text{const}$ in (4).

2. Let us now consider the directly opposite limiting case, when as $\tau \to \tau_0$, $p = \varepsilon$. All conceivable equations of state will represent an intermediate case between the two extremes considered ($p = 0,\ p = \varepsilon$). Let us note that the equation of state $p = \varepsilon$, proposed

first in (8), is a sufficiently accurate approximation to the equation of state of a real gas in the cores of superdense configurations (see (9)).

For the indicated equation of state, from (10), (11) it immediately follows that

$$X = e^{-\sigma} = C_1 E, \qquad Y = C_2 \rho^{4/3} \lambda^2 E. \tag{16}$$

For the energy density from (8) we have

$$E = -M_\lambda / 4\pi \lambda^2 \rho^2 \rho_\lambda, \tag{17}$$

and in place of equation (9) it is expedient to take

$$(M + M_\lambda \lambda)/M_\lambda = -(\rho + \lambda \rho_\lambda)/\rho_\lambda. \tag{18}$$

From the fact that M_λ is not equal to zero it follows that the gravitating mass inside R depends on the work of the pressure forces on the surface (5).

We shall seek M and r in the form

$$M = C_3 \lambda^m, \qquad r = C_4 \lambda^n, \tag{19}$$

where C_3, C_4, m, n are to be determined.

Equation (18) imposes on m and n the relation

$$n(m+1) = -m(1+n). \tag{20}$$

From (16), (17), and (19) we have

$$E = -\frac{C_3 m}{4\pi n C_4^3} \lambda^{m-3n-2}, \qquad X = -B\lambda^{m-3n-2}, \tag{21}$$

$$Y = -F\lambda^{m+n}, \qquad B = \frac{C_1 C_3 m}{4\pi n C_4^3}, \qquad F = C_2 C_3 C_4 \frac{m}{n}.$$

Substituting (21) into (7), we obtain

$$\frac{C_3}{C_4} \lambda^{m-n} = \left[1 - BC_4^2 n^2 \lambda^{m-n} + FC_4^2 (n+1)^2 \lambda^{m+3n}\right].$$

For the last equality to hold for variable λ, it is necessary that

$$m + 3n = 0, \qquad 1 = -FC_4^2(n+1)^2, \qquad C_3 = -BC_4^3 n^2. \tag{22}$$

In this case, from (20) and (22) we have

$$m = 1, \qquad n = -\frac{1}{3}, \qquad C_1 = 12\pi, \qquad 1 = \frac{4}{3}C_2 C_3 C_4^2. \tag{23}$$

Thus, the solution depends on two arbitrary constants, which, as in the case of the motion of matter with $p = 0$, in the general case corresponds to the greatest possible number of physically distinct arbitrary functions: the distribution of the density of matter and of radial velocities.

For the sought functions, with the aid of (23), (19), and (16), we have

$$m = \frac{aC_3R^2}{c(\tau_0 - \tau)}, \qquad \varepsilon = \frac{3C_3 a}{4\pi C_4^3(\tau_0-\tau)^2}, \qquad r = C_4 R\left[\frac{c(\tau_0-\tau)}{R}\right]^{1/3}, \tag{24}$$

$$e^{\omega} = \frac{1}{3C_2C_3C_4}\left[\frac{c(\tau_0-\tau)}{R}\right]^{2/3}, \qquad e^{\sigma} = \frac{3}{4\pi}\frac{C_1C_3}{C_4^3}.$$

Hence, as in the case of dust-like matter, as $\tau \to \tau_0$ the distribution of the energy density becomes homogeneous and tends to infinity as $(\tau_0 - \tau)^{-2}$, while the circumferential and radial distances

tend to zero as $\sim (\tau_0 - \tau)^{2/3}$, in contrast to dust-like matter, for which the indicated distances tend to zero as $\sim (\tau_0 - \tau)^{4/3}$ [4].

3. It appears possible, for $p = \varepsilon$, to construct another, more interesting class of particular self-similar solutions of the form (7^6, 10).

$$m = aC_1RM(t), \qquad r = C_2R\rho(t). \tag{25}$$

In this case, from (8) and (9) we have

$$M = \rho^{-1}, \qquad \rho = \varepsilon C^{-2} = AR^{-2}\rho^{-4}, \qquad A = C_1 a/4\pi C_2^3. \tag{26}$$

For $e^{-\sigma} = X$ and $e^{-\omega} = Y$, from (3), with the aid of (26), it follows that

$$X = \varphi(\tau)R^{-2}, \qquad Y = \psi = \text{const}. \tag{27}$$

Here $\varphi(\tau), \psi > 0$.

To determine ρ from (7), we have

$$\frac{C_1}{C_2 r^2} = (1 + \rho_\eta^2 - \psi\rho^2). \tag{28}$$

Here the dimensionless time $d\eta = C_2^2 c\, d\tau(\varphi(\tau))^{-1/2}$ has been introduced. From (28), for $z = \rho^2$, we have

$$\int \frac{dz}{\sqrt{b + \psi z^2 - z}} = 2\eta, \tag{29}$$

where $b = C_1 C_2^{-1}$, $\psi(R) > 0$.

From (29) two types of solution follow:

$$2(\eta - \eta_0) = \frac{1}{\sqrt{\psi}} \operatorname{Arsh} \frac{2\psi z - 1}{\sqrt{4\psi b - 1}} \qquad \text{for } 4\psi b > 1,$$

$$2(\eta - \eta_0) = \frac{1}{\sqrt{\psi}} \ln |2\psi z - 1| \qquad \text{for } 4\psi b = 1.$$

For the first of these, motions toward a point and away from a point are possible; for the second, an asymptotic approach from above or from below to the definite value $z_1 = 1/2\psi$.

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Note: Figure translations are in progress. See original paper for figures.