A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE WITH TWO LINES OF DEGENERATION

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Abstract

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A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE WITH TWO LINES OF DEGENERATION

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The paper investigates a boundary-value problem for the equation

$$y(y-1)u_{xx}+u_{yy}=0 \tag{L}$$

in a specially constructed mixed domain that contains within it intervals of the lines of degeneration $y=0,\ y=1$.

We introduce the following notation:

$$\omega(y)=\begin{cases}\dfrac{1}{4}(2y-1)\sqrt{y-y^2}+\dfrac{1}{8}\arcsin(2y-1), & \text{for } 0\le y\le 1,\\ \dfrac{1}{4}(2y-1)\sqrt{y^2-y}-\dfrac{1}{8}\ln\left|2y-1+2\sqrt{y^2-y}\right|, & \text{for } y\le 0,\ y\ge 1;\end{cases}$$

$$\tilde{x}=x/a,\qquad a=\text{const},\qquad \pi/8<a<\pi/4; \tag{1}$$

$$\tilde{y}(y)=\begin{cases}-(3/2a)^{2/3}(\omega\mp\pi/16)^{2/3}, & \text{for } 0\le y\le 1,\\ (3\omega/2a)^{2/3}, & \text{for } y\le 0,\ y\ge 1;\end{cases} \tag{2}$$

$$a\xi_1(x,y)=x+\omega-\pi/16,\qquad a\eta_1(x,y)=x-\omega+\pi/16;$$

$$b(\tilde{y})=[\tilde{y}'(y)]^{-2}\tilde{y}''(y),\qquad \tilde{\omega}(\tilde{y})=\exp\left(\frac{1}{2}\int_0^{\tilde{y}}b(s)\,ds\right). \tag{3}$$

In the plane of the variables x,y, consider the points $A_1(0,0)$, $A_0(\pi/8,0)$, $A_2(a,0)$, $A_3(a,1)$, $A_4(0,1)$, $A_5(a-\pi/8,0)$, $A(a-\pi/8,1)$, $B(a-\pi/16,1/2)$, $B_1(\pi/16,1/2)$.

Let D^* be the finite simply connected domain bounded by: 1) the arc $\sigma_0 (x - a/2)^2 + \omega^2 = a^2/4,\ y \le 0$; 2) the arc $\sigma_1 (x - a/2)^2 + \omega^2 = a^2/4,\ y \ge 1$; 3) the characteristics A_1B_1, $\eta_1 = \pi/8a$, B_1A_4, $\xi_1 = 0$, A_2B, $\xi_1 = 1 - \pi/8a$, BA_3, $\eta_1 = 1$, of equation (L). Further, let B_0 (D_0) be the point of intersection of the characteristics A_0B_0, $\eta_1 = \pi/4a$, and A_2B (AD_0, $\eta_1 = 1 - \pi/8a$, and B_1A_4); C the point of intersection of the characteristics A_0B_1 and A_3A_5; D the part of the domain D^* lying above the curve D_0AB; D^+ the elliptic part of the domain D; Δ (Δ^*) the hyperbolic part of D, situated above the curve A_4D_0A (ABA_3); D^- the domain of the characteristic quadrilateral $A_0B_0AD_0$.

The problem under study (Problem A) consists in finding a function $u(x, y)$ with the following properties:

1. $u \in C(\overline{D^*} - A_0 - A_2)$ and at the point A_0 (A_2) tends (may tend) to infinity of order $\alpha < 1/6$ (of logarithmic order);
2. u_y (u_x) is continuous everywhere in the closed domain $\overline{D^*}$, except for A_0 and, possibly, A, A_i $(i = 1, 2, \ldots, 5)$ and the characteristics issuing from them, where it tends to infinity of order $< 5/6$ ($< 7/6$) and < 1 (< 1), respectively;
3. u is a twice continuously differentiable solution of equation (L) everywhere in D^*, possibly except for the characteristics issuing from the points A_0, A, and A_5.
4. u satisfies the boundary conditions:

$$(\tilde\omega u)_{\sigma_1} = \varphi_1(\tilde x), \qquad 0 \le \tilde x \le 1; \tag{4}$$

$$(\tilde\omega u)_{A_4D_0} = \psi_1(\eta_1), \qquad 0 \le \eta_1 \le x_0; \tag{5}$$

$$(\tilde\omega u)_{A_3B} = \psi_2(\xi_1), \qquad x_0 \le \xi_1 \le 1; \tag{6}$$

$$(\tilde\omega u)_{BB_0} = \psi_3(\eta_1), \qquad 1 \le \eta_1 \le \pi/4a; \tag{7}$$

$$(\tilde\omega u)_{\sigma_0} = \varphi_0(\tilde x), \qquad 0 \le \tilde x < 1, \tag{8}$$

where $\tilde\omega$ is defined by formula (3), and in (2) the minus sign is taken before $\pi/16$, while $x_0 = 1 - \pi/8a$.

The following assumptions are made concerning the prescribed functions:

$$\varphi_1(\tilde x) \in C^2(0 < \tilde x < 1), \quad \varphi_1(\tilde x) = O(1)(\tilde x - \tilde x^2)^{\chi_1}, \quad \psi_1(\eta_1) \in C^7(0 < \eta_1 \le x_0),$$

$$\psi_1(\eta_1) = O(1)\eta_1^{\chi_2 - i}, \quad \psi_2(\xi_1) \in C^7(x_0 \leq \xi_1 < 1), \quad \psi_2^{(i)}(\xi_1) = O(1)(1-\xi_1)^{\chi_2 - 1},$$

$$5/6 < \chi_i = \text{const}, \quad i = 0, 1, \quad \psi_3(\eta_1) \in C^4(1 \leq \eta_1 \leq \pi/4a), \quad \psi_3(\eta_1) \in C^4$$

$$(1 < \eta < \pi/4a), \quad \psi_3'(\eta_1) = O\{(\pi/4a - \eta_1)^{-\alpha - 5/6}\}, \quad \varphi_0(x) \in C(0 \leq x < 1),$$

$$\varphi_0(\tilde{x}) \in C^2(0 < \tilde{x} < 1);$$

$\psi_3(\varphi_0)$ at the point $\eta_1 = 1$ $(\tilde{x} = 1)$ may tend to ∞ of order $\leq 2/3$ (logarithmic order).

Problem A cannot have more than one solution. This assertion is readily proved by the method of integral inequalities (^{1}, ^{2}), relying on the uniqueness of the solution of the Goursat problem for equation (L) in the domain D^- with data on the characteristics AD_0, AB_0. In doing so one must take into account the fact that the coefficient $y(y-1)$ for $0 \leq y \leq 1$ satisfies the condition of F. I. Frankl (^{3}), which ensures uniqueness of the solution of the Tricomi problem.

The proof of existence of a solution $u(x, y)$ of problem A consists of the following eight main stages:

1. In the domain D^+ a boundary-value problem of the Holmgren type (^{4}) is studied:

$$(\tilde{\omega}u)_{\sigma_1} = \varphi_1(\tilde{x}), \qquad \frac{\partial}{\partial y}(\tilde{\omega}u) = \nu(\tilde{x}) \text{ for } y = 1,\ 0 < \tilde{x} < 1,\ \tilde{x} \neq x_0.$$

2. In the domain Δ (Δ^*) a problem of the type of a singular Tricomi problem is considered, where the solution $u(x, y)$ of equation (L) is determined from the boundary condition (5) ((6)), knowing $\nu(\tilde{x})$ at the interior points of the segment A_4A (AA_3) of the straight line $y = 1$.
3. From the established constructive properties of the solutions of these two problems, the principal functional relations are derived between $\tau(\tilde{x}) = u(a\tilde{x}, 0)$ and $\nu(\tilde{x})$, carried from the elliptic and hyperbolic parts of the mixed domain D onto the line of degeneration $y = 1$.
4. In the domain D the existence of a solution $u(x, y)$ of the Hellerstedt problem with data (4)–(5)–(6) is proved. This is achieved by reducing it to an equivalent singular integral equation.
5. The behavior of the function u on AD_0 and AB is studied.

6. The Goursat problem is solved in the domain D^{-} with data on the characteristics AD_0 and ABB_0.
7. Let

$$(\tilde{\omega}u)_{A_0B_1} = \Psi'(\eta_1), \qquad \pi/8a \le \eta_1 < \pi/4a;$$

$$(\tilde{\omega}u)_{A_0B_0} = \Psi_2(\xi_1), \qquad 0 < \xi_1 \le x_0, \tag{9}$$

where $\tilde{\omega}$ is defined by formula (3), and in (2) before $\pi/16$, this time, the plus sign is taken.

It is established that:

a) $\Psi_1(\eta_1) \in C(\pi/8a \le \eta_1 < \pi/4a)$ and at the point A_0, $\eta_1 = \pi/4a$, has a singularity of order α with respect to $\pi/4a - \eta_1$.

b) $\Psi_1(\eta_1) \in C^4(\pi/8a \le \eta_1 < \pi/4a,\ \eta_1 \ne 1)$, and for $\eta_1 = 1$, $\Psi_1'(\eta_1)$ may tend to ∞, but of order not higher than 2/3.

c) $\Psi_2(\xi_1) \in C(0 < \xi_1 \le x_0)$ and at the point A_0 has a singularity of order α with respect to ξ_1.

g) $\Psi_2(\xi_1) \in C^4\ (0 < \xi_1 < x_0)$, and as $\xi_1 = x_0$, $\Psi_2'(\xi_1)$ tends, in general, to ∞ of order not exceeding 2/3.

8. In the domain $D^{*} \setminus (D \cup D^{-})$ the existence of a solution $u(x,y)$ of the Gellerstedt problem with data (8), (9) is proved.

The question of existence in the domain D of a solution $u(x,y)$ of the Gellerstedt problem (item 4) reduces to the following equivalent system of singular integral equations:

$$v_i(x) \mp \lambda \int_{-1}^{1} v_i(t) K_i(x,t)\, dt - \int_{-1}^{1} v_i(t) L_i(x,t)\, dt = h_i(x), \tag{10}$$

where $i = 1$ for $-1 < x < x^0$; $i = 2$ for $x^0 < x < 1$; $x^0 = 2x_0 - 1$; $v_1(x) = v_2(x) = v((x+1)/2)$, $\lambda = -1/\pi\sqrt{3}$; h_i, L_i are known functions;

$$K_i(x,t) = \left(\frac{1-(-1)^i t}{1-(-1)^i x}\right)^{2/3} \left(\frac{1}{t-x} + \frac{(-1)^i}{1-tx}\right).$$

Under the assumptions made above concerning the smoothness of the boundary data ψ_i, φ_1, the functions h_i will be at least such that $h_1 \in C^5\ (-1 < x \le x^0)$, $h_2 \in C^5\ (x^0 \le x < 1)$, $h_i(x) = O(1)((-1)^i - x)^{1/3}$.

The kernel $L_i(x,t)$ is regular. Moreover, it admits derivatives of arbitrary order in the domain of definition when $x \ne t$, while for $x = t$ the first derivatives tend, in general, to ∞ as $|x-t|^{-2/3}\ln|x-t|$; if the derivatives have a singularity, it can be separated explicitly at least in a neighborhood of the line $x=t$.

Equation (10), when $L_i \equiv 0$, was first studied by Gellerstedt in paper (2) (see also (5)). Using his results, the singular integral equation (10) can be reduced to an equivalent Fredholm equation of the second kind, whose unconditional solvability follows directly from the uniqueness of the problems formulated above.

The function $v_i(x)$, which is the solution of the integral equation (10), belongs to the class C^4 $(|x|<1,\ x \neq x^0)$ and at the points $x=\pm 1,\ x=x^0$ cannot tend to ∞ of order $>1/3$. The validity of this fact can be verified in the same way as in the case of the Tricomi problem (6).

After $v(\tilde{x})$ has been found, by solving the Holmgren problem in the domain D^+ and the singular Tricomi problem in the domains Δ (Δ^*) we construct a solution of problem A in the domain D.

The construction of the solution of problem A in the remaining part of the domain D^* is carried out according to the scheme proposed above (see (7)).

In conclusion we make the following remarks:

1. The degree of smoothness of the function Ψ_i is determined essentially by the smoothness of $v_i(x)$.
2. The singularity of order a of the solution $u(x,y)$ of problem A at the point A_0 is caused solely by the fact that the function $\psi_3'(\eta_1)$ at the point B_0 tends to ∞ of order $a+5/6$.
3. The function $u_y(x,0)$ at the point A_5 cannot tend to ∞ of order greater than $1/2$.
4. $u(x,0)$ at the point A_2 can have a logarithmic singularity only because $\dfrac{\partial}{\partial \xi_1}u(x,y)$ tends, in general, to ∞ of order $\leq 5/6$ along the entire characteristic AB_0.
5. The solution $u(x,y)$ of problem A in the case when the length a of the transition line is equal to $\pi/4$ is obtained from the case $a=\pi/4-\varepsilon$ by passage to the limit as $\varepsilon \to 0$. In this case $u(x,y)$ and its derivatives at the points A_1 and A_2 behave identically.

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REFERENCES

1. F. Tricomi, *On linear partial differential equations of second order of mixed type*, Moscow-Leningrad, 1947.

2. S. Gellerstedt, Ark. Math., Astr. och Fys., 26A, 3 (1936).

3. F. I. Frankl, PMM, 10, no. 3, 421 (1946).

4. E. Holmgren, Ark. Math., Astr. och Fys., 19B, 14 (1926).

5. A. R. Manwell, Arch. Rat. Mech. Analysis, 12, 3 (1963).

6. A. V. Bitsadze, *Equations of Mixed Type*, Publishing House of the Academy of Sciences of the USSR, 1959.

7. A. M. Nakhushev, DAN, 166, no. 3 (1966).

Note: Figure translations are in progress. See original paper for figures.

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