ON THE QUESTION OF THE INTERACTION OF SPINOR AND GRAVITATIONAL FIELDS

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ON THE QUESTION OF THE INTERACTION OF SPINOR AND GRAVITATIONAL FIELDS

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To establish the form of the interaction of an electron with the gravitational field, it is necessary to generalize the Dirac equation to the case in which the space-time continuum obeys Riemannian geometry, characterized by the interval

$$-ds^2 = g_{ik}dx^i dx^k. \tag{1}$$

This problem was first posed in the works ([1]) of Fock–Ivanenko. However, the Fock–Ivanenko method makes essential use of the formalism of orthogonal tetrads, whereas the tetrad components $h^{(a)}_{\cdot k}$ are not uniquely determined by Einstein' s gravitational equations, which are formulated in terms of the components of the metric tensor and their first and second derivatives with respect to the coordinates. For given g_{ik}, the tetrad components can be determined only up to a local Lorentz transformation ([2])

$$'h^{(a)}_{\cdot k} = L^{(a)}_{\cdot(b)} h^{(b)}_{\cdot k}, \tag{2}$$

where $L^{(a)}_{\cdot(b)}(x)$ is a pseudo-orthogonal matrix, and the Fock–Ivanenko equations are not invariant with respect to the transformation (2).

In connection with the foregoing, the problem arises of finding such a generally covariant formulation of the Dirac equations in which only the components of the metric tensor are used. The present work is devoted to the solution of this problem.

We shall proceed from the equation ([3])

$$g^{ik} P_{iP} k + m^2 c^2 = 0, \tag{3}$$

which relates the components of the 4-momentum of a particle in a gravitational field.

The transition from equation (3) to the generally covariant generalization of the Klein–Gordon equation can be carried out by replacing, in (3), the 4-momentum by the operator

$$\hat{P}_k = \frac{\hbar}{i}\nabla_k, \tag{4}$$

where ∇_k is the symbol of the covariant derivative with respect to the Christoffel symbols, followed by application of the obtained operator equation to the scalar wave function ψ. In this way we arrive at the equation

$$g^{ik}\nabla_i\nabla_k\psi - \frac{m^2c^2}{\hbar^2}\psi = 0, \tag{5}$$

which is the direct generalization of the nonrelativistic Schrödinger equation. The validity of the last statement is easily verified by considering the gravitational field to be weak,

$$g^{00} = -\left(1 + \frac{2U}{c^2} + O\left(\frac{1}{c^3}\right)\right), \qquad g^{0\alpha} = O\left(\frac{1}{c^3}\right), \qquad g^{\alpha\beta} = \left(1 - \frac{2U}{c^2} + O\left(\frac{1}{c^3}\right)\right) \tag{6}$$

and setting

$$\psi = e^{-imcx^0/\hbar}\,\mathring{\psi}; \tag{7}$$

substituting these values into equation (5) and passing to the limit as $c \to \infty$, we obtain the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial\mathring{\psi}}{\partial t} + \frac{\hbar^2}{2m}\Delta\mathring{\psi} + mU\mathring{\psi} = 0. \tag{8}$$

However, equation (5) is suitable for describing only spinless particles and, consequently, is inapplicable to the electron.

To obtain a wave equation that takes particle spin into account, we transform equation (3), representing it in the form

$$\left(\gamma^k P_k - imcI\right)\left(\gamma^l P_l + imcI\right) = 0, \tag{9}$$

where γ^k are matrices satisfying the relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}I, \tag{10}$$

and I is the unit matrix.

It follows from equation (9) that, by its tensor dimension, the matrix γ^i is a contravariant vector, while I is a scalar, whence follow the rules for covariant differentiation of the γ-matrices:

$$\nabla_k \gamma^i = \partial_k \gamma^i + \Gamma^i_{kl} \gamma^l. \tag{11}$$

Equation (10) determines the matrices γ^i only up to a transformation ([4])

$$'\gamma^i = S^{-1} \gamma^i S, \tag{12}$$

where $S(x)$ is some arbitrary matrix. The quantities γ^i should then be chosen in such a way that the condition

$$\nabla_k \gamma^k = 0. \tag{13}$$

is satisfied.

Returning to equation (9), we see that it can be satisfied by setting

$$\gamma^k P_k - imcI = 0, \tag{14}$$

whence, with the aid of substitution (4), we obtain the equation

$$\frac{\hbar}{i} \gamma^k \nabla_k \psi - imc\psi = 0. \tag{15}$$

It remains to clarify the meaning of the symbol $\nabla_k \psi$, by establishing the transformation properties of the spinor ψ in Riemannian space. Following Sommerfeld ([5]), we shall define a spinor in Riemannian space as a geometric object whose bilinear combination of the form

$$A^i = \bar{\psi} \gamma^i \psi, \tag{16}$$

transforms as a contravariant vector.

It follows from this definition that a spinor is a one-row matrix with scalar elements. Therefore, according to (11),

$$\nabla_k \psi = \partial_k \psi. \tag{17}$$

Together with (17), equation (15) is the desired generally covariant generalization of the Dirac equation.

Applying to equation (15) the operator

$$\frac{\hbar}{i}\gamma^l\nabla_l + imcI, \tag{18}$$

and taking into account formulas (10), (11), and (17), we arrive at the equation

$$g^{ik}\nabla_i\nabla_k\psi - \frac{m^2c^2}{\hbar^2}\psi + \left(\gamma^i\nabla_i\gamma^k\right)\nabla_k\psi = 0, \tag{19}$$

which contains a typically spin term (the third term).

For the generally covariant formulation of the Dirac equation, another approach is also possible, connected with the abandonment of the symmetry of the time and space coordinates. In this case equation (3) can be represented in the form

$$(\sqrt{-g^{00}} - h^\lambda P_\lambda)^2 = h^{\mu\nu}P_\mu P_\nu + m^2c^2, \tag{20}$$

where

$$h^\lambda = g^{0\lambda}/\sqrt{-g^{00}}, \qquad h^{\mu\nu} = g^{\mu\nu} - g^{0\mu}g^{0\nu}/g^{00}. \tag{21}$$

The three-dimensional tensors $h^\lambda, h^{\mu\nu}$ are chronometrically [6] invariant.

Using chronometrically invariant matrices α^μ and α, subject to the relations

$$\alpha^\mu\alpha^\nu + \alpha^\nu\alpha^\mu = 2h^{\mu\nu}I, \qquad \alpha^\lambda\alpha + \alpha\alpha^\lambda = 0, \qquad \alpha^2 = I, \tag{22}$$

and, moreover,

$$\gamma^0 = i\sqrt{-g^{00}}\alpha, \qquad \gamma^\mu = \alpha^\mu - ih^\mu\alpha, \tag{23}$$

the right-hand side of equation (20) can be written as a complete square, so that

$$\sqrt{-g^{00}}\,\frac{E}{c} = -h^\lambda P_\lambda \pm (\alpha^\mu P_\mu + \alpha mc) \qquad (E = -P_0c). \tag{24}$$

Replacing in (24) E, P^μ by the chronometrically invariant differential operators

$$\hat{E} = -\frac{\hbar}{i}\partial_0^* \equiv -\frac{\hbar}{i}\,\frac{c}{\sqrt{-g_{00}}}\partial_0, \tag{25}$$

$$\hat{P}_\mu = \frac{\hbar}{i}\partial_\mu^* = \frac{\hbar}{i}\left(\partial_\mu - \frac{g_{0\mu}}{g_{00}}\partial_0\right), \tag{26}$$

we arrive at the equation

$$\frac{1}{c}\sqrt{-g^{00}}\partial_0^*\psi = h^\mu \partial_\mu^*\psi \pm \left(\alpha^\mu \partial_\mu^*\psi + i\alpha \frac{mc}{\hbar}\psi\right), \tag{27}$$

where the spinor ψ is still regarded as a scalar single-column matrix.

From equation (27) follows an asymmetry of the masses of particles and antiparticles in a gravitational field [7]. However, this asymmetry depends essentially on the choice of the reference frame. Thus, for example, a nucleon and an antinucleon at rest with respect to a rotating reference frame possess different rest masses.

Since the general gravitational field is inhomogeneous and is not described by any transformation groups, it is known that the conservation law of energy-momentum is not fulfilled in it. We now come to the conclusion that in a gravitational field the concept of spin as a quantum number loses its meaning, and the principle of combined inversion ceases to be an absolute law of nature; it is fulfilled only in flat 4-space. In curved 4-space, in different reference frames it will be violated in different ways and will hold only in certain preselected reference frames, where $g_{0\alpha} = 0$.

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Note: Figure translations are in progress. See original paper for figures.