SUBGROUPS OF PERIODIC GROUPS

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S. P. STRUNKOV

SUBGROUPS OF PERIODIC GROUPS

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In the present paper we consider a class of infinite periodic groups which includes all locally finite groups. These are groups in which any two elements generate a finite subgroup (in the paper they are called $F2$-groups). It turns out that every $F2$-group contains an infinite abelian subgroup, whence there follows the analogous theorem for locally finite groups proved by M. I. Kargapolov (^{7}). Thus the well-known problem of O. Yu. Schmidt on the existence of an infinite proper subgroup in an infinite noncommutative group is reduced to the analogous question for groups with two generators. We shall call the following question the generalized O. Yu. Schmidt problem: does a group contain a proper subgroup equipotent to it? It turns out that for uncountable $F2$-groups whose cardinality is regular, the generalized Schmidt problem has a positive solution. In conclusion, some conditions are considered under which $F2$-groups are locally finite. In particular, the question of local finiteness of $F2$-groups with the minimal condition for subgroups is reduced to the well-known problem of S. N. Chernikov on the extremality of locally finite groups with the minimal condition for subgroups.

Lemma 1. *If there exists an infinite noncommutative group all of whose proper subgroups have cardinality less than the cardinality of the whole group, then this group is an extension of its center by means of a simple group.*

Thus Lemma 1 reduces Schmidt's problem and the generalized Schmidt problem to simple groups.

Lemma 2. *Let a certain class Σ of infinite groups be given, closed with respect to taking infinite subgroups and quotient groups by finite normal divisors. Then the following assertions are equivalent: 1) in every group $G \in \Sigma$ there exists at least one element whose normalizer is infinite; 2) every group $G \in \Sigma$ contains an infinite abelian subgroup.*

An elegant proof of this result, very useful for us, is given in (^{2}).

We shall say that a group G has property $N1$ if it contains at least one nonidentity element whose normalizer has cardinality $\mu(G)$ ($\mu(G)$ is the cardinality of the group G). We shall denote by $D_a(x)$ the set of elements

$x, a^{-1}xa, a^{-2}xa^2, \dots, a^{1-n}xa^{n-1}$, where n is the order of the element a. The set $D_a(x)$ is called **abelian** if all its elements commute with one another.

Lemma 3. *Suppose that in some conjugacy class of elements of a group G of regular cardinality $\mu(G)$ there exists a set L of cardinality $\mu(G)$ of elements x_α such that all the sets $D_a(x_\alpha)$ are abelian. Then the group G has property $N1$.*

Recall that a cardinal number ω is called **regular** if it cannot be represented as a sum in which the number of summands is less than ω, and each summand is also less than ω (^{9}).

Using the Thompson–Feit theorem on the solvability of groups of odd order (^{4}), one can prove the following assertion.

Lemma 4. *Let in a group G of regular cardinality $\mu(G)$ there exist a set of finite subgroups H_α of cardinality $\mu(G)$, containing some element a, and such that the cardinality of the set of conjugacy classes of elements containing all the subgroups H_α is less than $\mu(G)$. Then the group G has property $N1$.*

In what follows we shall consider groups in which any two elements generate a finite subgroup. For brevity we shall call such groups $F2$-groups. The class of $F2$-groups is broader than the class of locally finite groups. As shown in (^{3}), there exist non-locally finite $F2$-groups which are even p-groups. It is easy to see that the class of $F2$-groups is closed under taking infinite subgroups and infinite factor groups. Consequently, Lemma 2 holds for $F2$-groups.

Lemma 5. *If in an $F2$-group G of regular cardinality $\mu(G)$ the cardinality of the set of conjugacy classes of elements of a given order n is equal to $\mu(G)$, then the group G has property $N1$.*

Theorem 1. *An uncountable $F2$-group G of any regular cardinality $\mu(G)$ has a proper subgroup of cardinality $\mu(G)$ with nontrivial center.*

Since every cardinal number of uncountable index is regular, Theorem 1 implies

Corollary. *If the cardinality of every proper subgroup of an $F2$-group G is at most ω (where ω is some infinite cardinal), then the cardinality of the group G is also at most ω.*

We shall now be interested in countable $F2$-groups.

Lemma 6. *Let Π be a set of natural numbers which are the orders of elements of some countable $F2$-group G. Then, if together with some number m the set Π contains infinitely many of its multiples, the group G has property $N1$.*

Lemma 7. *If the orders of all finite p-subgroups of a countable $F2$-group G, for some prime number p, are unbounded in the aggregate, then the group G has property $N1$.*

From the results of Higman (^{5}), Brauer (^{1}), and Chunikhin' s theorem on the existence and conjugacy of Π-subgroups in a Π-solvable finite group (^{6}), it follows that

Lemma 8. *If there exist infinitely many noncommutative simple finite pairwise nonisomorphic groups P_i which are subgroups or factor groups of some finite subgroups of a countable $F2$-group G, then the group G has property $N1$.*

From Lemmas 3–8 it follows that property $N1$ is present in every $F2$-group. From Lemma 2 it now follows that

Theorem 2. *If in a group any two elements generate a finite subgroup, then this group contains an infinite abelian subgroup.*

This theorem generalizes the well-known theorem of M. I. Kargapolov ([7]) on the existence of an infinite abelian subgroup in an infinite locally finite group, and is proved without using that result. B. H. Neumann and Kovács constructed an example of an uncountable locally finite group all of whose abelian subgroups are at most countable. Consequently, in this sense Theorem 2 does not admit a generalization to groups of arbitrary cardinality. In view of Theorem 2, the Schmidt problem formulated above reduces to an analogous question for groups with two generators. Namely, there holds

Corollary. *If the minimal number of generators of a noncommutative group is greater than two, then the group contains an infinite proper subgroup.*

In conclusion we shall consider some conditions under which $F2$-groups are locally finite. Recall that a group is called **ex-**

extremal if it contains an abelian subgroup of finite index. It is known that if a periodic group is extremal, then it is locally finite.

Let an $F2$-group G satisfy the minimality condition for abelian subgroups. Then, by Theorem 2, it contains a subgroup K of type p^∞. It is easy to see that every subgroup $\{g, K\}$ of the $F2$-group G is locally finite.

Lemma 9. *If in an $F2$-group G with the minimality condition for abelian subgroups every subgroup $\{g, K\}$, generated by an arbitrary element g and an arbitrary quasicyclic subgroup K, is extremal, then the group G itself is extremal (and consequently locally finite).*

Thus the question of whether an $F2$-group with the minimality condition for subgroups is locally finite reduces to a known problem of S. N. Chernikov for locally finite groups (even for groups of the form $\{g, K\}$): is a locally finite group with the minimality condition for subgroups extremal (([8] 2.1)?

From the Thompson–Feit theorem ([4]), Lemma 9, and S. N. Chernikov's theorem on the extremality of locally soluble groups with the minimality condition for abelian subgroups ([8]), it easily follows that

Theorem 3. *If an $F2$-group with the minimality condition for abelian subgroups contains no elements of order $6a$, then it is locally finite.*

In exactly the same way, using the known theorem of Burnside on the solubility of finite groups of order $p^\alpha q^\beta$, one can obtain the following result.

Theorem 4. *If the set of prime divisors of the orders of elements of an $F2$-group with the minimality condition for abelian subgroups consists of two numbers, then such a group is locally finite.*

We note that an analogous theorem for p-groups was proved by O. Yu. Schmidt (([11], p. 298).

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REFERENCES

[1] International Mathematical Congress in Amsterdam, 1954, Moscow, 1961.
[2] P. Hall, C. R. Kulatilaka, *J. Lond. Math. Soc.*, **39**, 235 (1964).
[3] E. S. Golod, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **28**, No. 2, 273 (1964).
[4] W. Feit, J. Thompson, *Proc. Nat. Acad. Sci. U.S.A.*, **48**, No. 6 (1962).
[5] G. Higman, *J. Lond. Math. Soc.*, **32**, No. 3, 335 (1957).
[6] S. A. Chunikhin, *Dokl. Akad. Nauk*, **73**, No. 1, 29 (1950).
[7] M. I. Kargapolov, *Siberian Mathematical Journal*, **4**, No. 1, 232 (1963).
[8] S. N. Chernikov, *Uspekhi Mat. Nauk*, **14**, 5, 45 (1959).
[9] P. S. Aleksandrov, *Introduction to General Theory of Sets and Functions*, Moscow-Leningrad, 1948.
[10] M. Hall, *The Theory of Groups*, Moscow, 1962.
[11] O. Yu. Schmidt, *Selected Works*, **1**, Moscow, 1959.

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