EQUIAFFINE GEOMETRY OF A DUAL FIELD AND ITS APPLICATION TO THE THEORY OF STEADY FLUID FLOW

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Abstract

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EQUIAFFINE GEOMETRY OF A DUAL FIELD AND ITS APPLICATION TO THE THEORY OF STEADY FLUID FLOW

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1. We shall call a **dual field** a geometric object whose element is a pair ([1]): a vector $\mathbf{v}$ and a covector V, assigned at each point A of some domain $\mathcal{G}$ of three-dimensional space, with $(\mathbf{v}V) = 1$. Since we shall use Cartan' s method of exterior forms, all functions defining the dual field $\{\mathbf{v}, V\}$ are assumed to be analytic. Geometrically, a vector is an ordered pair of points, and a covector is an ordered pair of parallel planes. The scalar product of a vector by a covector is understood in the sense of the product of tensors with contraction. At each point A of the domain $\mathcal{G}$ we assign an equiaffine frame $R\{A, e_i\}$ and the reciprocal frame $R^*\{A, E^i\}$, where E^i are covectors and $(e^j_{iE}) = \delta^j_i$. All Latin indices here and below take the values $1, 2, 3$, and all Greek indices $1, 2$; summation is performed over repeated indices. The derivation formulas for the frames R and R^* have the form

$$(R) \quad dA = \omega^i e_i, \qquad de_j = \omega^i_j e_i.$$

$$(R^*) \quad dE^i = -\omega^i_j E^j.$$

The Pfaff forms ω^i, ω^i_j are related by the structure equations $D\omega^i = [\omega^j \omega^i_j]$, $D\omega^i_j = [\omega^k_j \omega^i_k]$ and by the equiaffinity condition $\omega^i_i = 0$. If a field of covectors $\{\mathbf{M}\}$ $(\mathbf{M} = \mu^i_{iE})$ is given, then the nonholonomic surface $(dA, \mathbf{M}) \equiv \mu_i \omega^i = 0$ will be called the **base of the field of covectors** $\{\mathbf{M}\}$.

Let us include the element in the frame, putting $\mathbf{v} = e_3$, $V = E^3$. The forms $\omega^i, \omega^i_3, \omega^3_i$ then become principal. Assuming the domain $\mathcal{G}$ to be three-dimensional and choosing the forms ω^i as basic, we write the fundamental system ([1]) of Pfaff equations in the form

$$\omega^i_3 = \Gamma^i_{3k} \omega^k, \qquad \omega^3_i = \Gamma^3_{ik} \omega^k. \tag{1}$$

The quantities Γ^i_{3k} and Γ^3_{ik} form tensors. The tensor Γ^i_{3k} is $\operatorname{grad}\mathbf{v}$, and the tensor Γ^3_{ik} is equal to $-\operatorname{grad} V$ ([2]). We also note that $\operatorname{div}\mathbf{v} = \Gamma^i_{3i}$. From (1) one obtains a series of invariant vectors, covectors, and tensors of the dual field. The vector

$$\vec{\tau} = \Gamma^i_{33} e_i \quad \text{or} \quad \vec{\tau} = (\mathbf{v}, \operatorname{grad}\mathbf{v}) \tag{2}$$

determines the direction of the tangent to the hodograph of the vector $\mathbf{v}$ when the latter is displaced along the streamline $\omega^1 = \omega^2 = 0$. The vector $\mathbf{s} = \Delta_{23}e_1 + \Delta_{31}e_2 + \Delta_{12}e_3$, where

$$\Delta_{ik} = \begin{vmatrix} \Gamma^1_{3i} & \Gamma^1_{3k} \\ \Gamma^2_{3i} & \Gamma^2_{3k} \end{vmatrix},$$

determines the direction of the tangent to an "equidirectional" line, i.e., a line along which $d\mathbf{v} \parallel \mathbf{v}$ and the vector $\mathbf{v}$ is displaced parallel to itself. In addition, consider the vector $\vec{\omega} = \operatorname{rot} V$ and the tensors $\Gamma^\alpha_{3\beta}$ and $\Gamma^3_{\alpha\beta}$.

Theorem 1. *If* $(\mathbf{v}, \vec{\tau}, \mathbf{s}) = 0$, *then the complex* (K) *of lines* $\mathbf{r} = A + \lambda\mathbf{v}$ *is special* ([3]).

Theorem 2. If $s = 0$, then there exists a family of "equidirectional" holonomic surfaces, i.e., surfaces along which $dv \parallel v$. In this case the complex (K) is cylindrical, i.e., it is a one-parameter family of bundles of parallel straight lines ([3]).

Theorem 3. If $\Gamma^\alpha_{3\beta} = 0$, then $s = 0$, and the base of the covector field $\{V\}$ is a family of equidirectional surfaces.

Theorem 4. If $\Gamma^3_{\alpha\beta} = 0$, then the base of the covector field $\{V\}$ is a bundle of planes.

2. A steady flow of a fluid whose velocity field $\{v\}$ has a family of equidirectional surfaces will be called **cylindrical**, since the associated complex (K) of its velocity field is cylindrical. The mathematical model of such a flow is a double field for which $\Gamma^\alpha_{3\beta} = 0$. The covector V is then completely determined at each point A by the specification of the vector v. A unit vector field having a family of equidirectional surfaces was first considered by S. S. Byushgens ([4]). The family of equidirectional surfaces (the base of the covector field $\{V\}$) will be called the **base of the cylindrical flow**. The vector $\omega = \operatorname{rot} V$ will be called the **affine vorticity vector of the cylindrical flow**. The vector $\vec{\tau}$, as follows from (2), is the acceleration (see (2)).

Theorem 5. A cylindrical flow exists with an arbitrary choice of two functions of three arguments. The equidirectional surfaces may be arbitrary surfaces, and the streamlines arbitrary curves.

Theorem 6. The divergence of the velocity field of a cylindrical fluid flow at a point A is equal to the projection of the acceleration vector $\vec{\tau}$ onto the velocity vector v parallel to the tangent plane of the base. If, however, the fluid is incompressible, then $\vec{\tau} \parallel V$.

Theorem 7. The associated vector ([5]), $p = \vec{\tau} - (\operatorname{div} v)v$, of a cylindrical flow always lies in the tangent plane of the base. If at every point A it is tangent to an asymptotic line of the base, then these asymptotic lines are straight, and the base is a family of cylindroids.

If, under displacement along each equidirectional surface, $dv = 0$, i.e., the velocity vector changes neither in direction nor in magnitude, then such equidirectional surfaces will be called **surfaces of translation**. A cylindrical flow whose base is a family of surfaces of translation will be called **instantaneously parallel**.

Theorem 8. An instantaneously parallel flow exists with an arbitrary choice of one function of three arguments. It is completely determined by specifying, in a fixed coordinate system, an arbitrary surface (one of the surfaces of translation) and an arbitrary line L (one of the streamlines) with a one-dimensional double field $\{v, V\}$ along L.

Theorem 9. An arbitrary surface (realized, for example, in the form of a thin film) can move, undergoing compression or stretching, in such a way that at every instant all its points have the same velocity.

If $\vec{\omega} = 0$, then the flow will be called **affine-irrotational**. The covector field $\{V\}$ is in this case the gradient of some scalar function: $V = \operatorname{grad} \varphi$.

Theorem 10. The velocity vector of an affine-irrotational instantaneously parallel flow is parallel to the affine normal ([6]) of the base.

Theorem 11. If the vector p of an affine-irrotational instantaneously parallel flow is tangent to an asymptotic line of the base, then either the streamlines are plane, or the base of the flow is a bundle of planes.

If the fluid is incompressible, then $\operatorname{div} \mathbf{v} = 0$. In this case, everything stated in Theorems 7 and 11 with respect to the vector $\mathbf{p}$ is valid with respect to the acceleration vector $\vec{\tau}$.

Theorem 12. *The base of an instantaneously parallel flow of an incompressible fluid is a family of cylinders with generators*

$$\mathbf{r} = \mathbf{A} + \lambda \vec{\tau}.$$

Theorem 13. *In an affine irrotational instantaneously parallel flow of an incompressible fluid, either the streamlines are plane curves, or the base of the flow is a pencil of planes.*

A cylindrical flow with plane streamlines will be called a **generalized plane-parallel** flow. In such and only in such a flow, the velocity vector $\mathbf{v}$ at each point A is parallel to one and the same plane. The vector $\vec{\tau}$ at each point A is parallel to this same plane. The streamlines of a generalized plane-parallel flow lie in a family of parallel planes, and the flow in each plane is obtained from the flow in one of them by some affine transformation. Special cases of a generalized plane-parallel flow are noted in Theorems 11 and 13, as well as in work (7).

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Note: Figure translations are in progress. See original paper for figures.