# THE TRICOMI PROBLEM FOR AN ELLIPTIC-HYPERBOLIC EULER–POISSON–DARBOUX EQUATION

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## Abstract

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# THE TRICOMI PROBLEM FOR AN ELLIPTIC-HYPERBOLIC EULER–POISSON–DARBOUX EQUATION

*(Presented by Academician I. N. Vekua on 1 III 1966)*

In the present paper we consider a boundary-value problem of Tricomi type for the following "model" equations of mixed type

$$\operatorname{sgn} y\,|y|^{\gamma}u_{xx}+u_{yy}+\frac{\alpha}{y}u_y=0, \tag{1}$$

$$\operatorname{sgn} y\,|y|^{\gamma}\omega_{xx}+\omega_{yy}+\frac{\beta}{y^2}\omega=0, \tag{1'}$$

where $\gamma, \alpha, \beta$ are constants, which we shall subject to certain conditions. Representing $\beta$ in the form $\beta=\frac{\alpha}{2}\{1-(\frac{\alpha}{2})\}$, so that

$$\alpha=1-\sqrt{1-4\beta} \tag{2}$$

(with the additional condition $\alpha<1$), and making in $(1')$ the substitution

$$\omega=|y|^{\alpha/2}u, \tag{3}$$

we arrive at equation (1). Therefore it is sufficient to study only equation (1).

We indicate results connected with equations (1) and $(1')$.

In the monograph of I. N. Vekua ([1]) the characteristic equation of infinitesimal bendings of surfaces of revolution is derived (p. 506, relations (11.16), (11.21), (11.22)). The corresponding "model" equation can be written in the form

$$\operatorname{sgn} y\,\omega_{xx}+\omega_{yy}-\frac{7}{36}\frac{\omega}{y^2}=0. \tag{4}$$

This equation is a special case of $(1')$ for $\gamma=0,\ \beta=-7/36$.

Many studies are devoted to the investigation of special cases of equation (1). For $\gamma = 0,\ y > 0$, equation (1) coincides with the well-known Euler—Poisson—Darboux equation (see, for example, ([2])). In view of this, we shall call (1) an elliptic-hyperbolic Euler—Poisson—Darboux equation.

In the work of I. L. Karol' ([3]) a special case of equation (1) in a mixed elliptic-hyperbolic domain is considered. Namely, a problem of Tricomi type is studied there for the equation

$$u_{xx} + yu_{yy} + \alpha u_y = 0 \tag{5}$$

for

$$0 < \alpha < 1. \tag{6}$$

Since in this problem the required function $u(x, y)$ satisfies equation (5) for $y \neq 0$, it can be reduced to the form

$$y^{-1}u_{xx} + u_{yy} + \frac{\alpha}{y}u_y = 0, \tag{7}$$

i.e. to equation (1), for $\gamma = -1$.

We shall extend the indicated result of I. L. Karol' to the general case of equation (1) under the following additional conditions:

$$-2 < \gamma, \tag{8}$$

$$-(\gamma + 1) < \alpha < 1. \tag{9}$$

For equation (1′), condition (9) is replaced by the condition

$$\frac{1}{4} - \frac{(\gamma + 2)^2}{4} < \beta < \frac{1}{4}, \tag{9'}$$

which is satisfied for equation (4).

**Statement of the problem.** Let $\Omega$ be the domain bounded by: a simple curve $\Gamma$, lying entirely in the half-plane $y > 0$ and resting on the segment $AB$ of the axis $y = 0$, which, without loss of generality, we assume to coincide with the interval $[0, 1]$ of the $x$-axis; and by two characteristics of equation (1), of different families, $AC$ and $BC$, issuing from the points $A$ and $B$ and intersecting at the point $C$:

$$AC: \quad x - \frac{2}{\gamma+2}(-y)^{(\gamma+2)/2} = 0, \tag{10}$$

$$BC: \quad x + \frac{2}{\gamma+2}(-y)^{(\gamma+2)/2} = 1. \tag{11}$$

The part of the domain $\Omega$ lying in the half-plane $y > 0$ ($y < 0$) will be denoted by $\Omega_+$ ($\Omega_-$).

**Problem $T_\alpha$.** Find a function $u(x,y)$, continuous in $\Omega$, such that:

1) In $\Omega_+$, the function $u(x,y)$ is a twice continuously differentiable solution of equation (1).
2) In $\Omega_-$, the function $u(x,y)$ is a generalized solution of equation (1), which will be defined below.
3) The derivative $u_x$ of the solution is continuous at the interior points of the segments $AB$ of the line of degeneration, and the derivative $u_y$ satisfies there the gluing condition

$$\left[|y|^\alpha u_y\right]^+ = \left[|y|^\alpha u_y\right]^- = \nu(x), \tag{12}$$

where $[\,]^+$ ($[\,]^-$) denotes the limit from $\Omega_+$ ($\Omega_-$), and $\nu(x)$ is a function continuous on $(0,1)$, in general not identically equal to zero.

4) Boundary values of $u(x,y)$ are prescribed on the unclosed contour $\Gamma + AC$:

$$u|_\Gamma = \varphi(s), \quad 0 \le s \le l \quad (s \text{ is the arc of } \Gamma), \tag{13}$$

$$u|_{AC} = \psi(x), \quad 0 \le x \le \frac{1}{2}, \tag{14}$$

where $\varphi(s)$ and $\psi(x)$ are prescribed functions, continuous on $[0,l]$ and $[0,\frac{1}{2}]$, respectively, and satisfying certain additional conditions, which will be stated below.

Taking into account the substitution (3), it is easy to formulate an analogous problem for equation (1′). The essential difference here is that the continuity condition for the solution on $AB$ and condition (12) are replaced by the following:

$$\left[|y|^{-\alpha/2}\omega\right]^+ = \left[|y|^{-\alpha/2}\omega\right]^-, \tag{15}$$

$$\left[|y|^\alpha\left(|y|^{-\alpha/2}\omega\right)'_y\right]^+ = \left[|y|^\alpha\left(|y|^{-\alpha/2}\omega\right)'_y\right]^-, \tag{16}$$

where $\alpha = 1 - \sqrt{1-4\beta}$. In particular, for equation (4): $\alpha = -\frac{1}{3}$.

Let us note that some special cases of equation (1) (or equations easily reducible to it) are available in [4, 5], p. 220; [6, 7]. All the problems considered there fall within the range of questions of problem $T_\alpha$ and therefore can be investigated by the method proposed below.

Problem $T_\alpha$ is reduced to well-studied problems by means of the following one-to-one and continuous change of variables:

$$x = x, \quad t = \operatorname{sgn} y\,(1-\alpha)^{2(\alpha-1)/(\gamma+2)}|y|^{1-\alpha} \tag{17}$$

$$(t = 0 \text{ when } y = 0).$$

This substitution is given in (3) for the case $\gamma = -1$.* Substitution (17) has the following properties.

1°. It transforms equation (1) into the simpler model equation of mixed type

$$\operatorname{sgn} t\,|t|^m u_{xx} + u_{tt} = 0, \tag{18}$$

where $m = (\gamma + 2\alpha)/(1-\alpha)$, and, in view of (8) and (9),

$$-1 < m. \tag{19}$$

2°. It transforms the gluing condition (12) into the condition of continuity of the derivative $u_t(x,t)$ at the interior points of the segment $AB$.

3°. It transforms the characteristics of equation (1) into the characteristics of equation (18), preserving the membership of a characteristic in a definite family. In particular, the characteristic triangle $ABC$ is transformed into the characteristic triangle $ABC^*$, where $C^*$ is the image of $C$.

4°. It transforms the "normal curve" of equation (1)

$$(x-1/2)^2 + \frac{4}{(\gamma+2)^2}\,y^{\gamma+2} = 1/4$$

into the "normal curve" of equation (18).

Taking these properties into account, we arrive at the conclusion that the problem $T_\alpha$ posed above corresponds in the $(x,t)$-plane to the well-known Tricomi problem (problem $T$) for equation (18). (The formulation of problem $T$ is obtained from the formulation of problem $T_\alpha$ for $\alpha = 0$.)

The question of existence and uniqueness of a solution of problem $T$ for equation (18), under various assumptions concerning the class of solutions, the curve $\Gamma^*$

(the image of Γ), and the boundary data, has been the subject of numerous investigations. A detailed bibliography is available in ([5],[8] ).

Usually the solution $u(x,t)$ of problem $T$ for $t<0$ is sought in a certain class of generalized solutions. Passing to the variables $(x,y)$, we shall obtain a certain class of generalized solutions of problem $T_\alpha$ for $y<0$. Of course, in determining restrictions on the curve Γ and on the functions $\varphi(s)$ and $\psi(x)$, we must proceed from the requirement that, in the $(x,t)$-plane, conditions be obtained that ensure uniqueness and existence of a solution of problem $T$ in the chosen class.

Without going into details, we indicate that, when solutions are sought in the most general classes, one may use, for $m<0$, the results of ([9]), and for $m>0$, the results of ([10])** (Ch. V, § 5, Lemma 4). It is easy to obtain that in this case the functions $\psi(x)$ and $\varphi(s)$ must satisfy the following additional conditions. The function $\psi(x)$ has continuous derivatives of the first and second orders on $[0,1/2]$. For $m<0$, the function $\varphi(s)$ has a continuous derivative of the first order on $[0,l]$ and admits, in a neighborhood of the points $A$ and $B$, the representation

$$\varphi(s)=y^{2+\gamma}\,\bar{\varphi}(x), \tag{20}$$

where $\bar{\varphi}(x)$ is a function continuous on $[0,1]$. To obtain similar conditions when choosing a narrower class of solutions, additional investigations in a neighborhood of the points $A$ and $B$ are required.

Finally, let us note the following. If $\Gamma^*$ is a normal curve, then the solution of problem $T$ for equation (18) is found in quadratures. In view of property 4°, we obtain the same result for problem $T_\alpha$ as well, if Γ is a normal curve.

**Remark.** Starting from various generalizations of problem $T$, one can investigate, by means of substitution (17), various generalizations of problem $T_\alpha$. In exactly the same way one can also investigate other types of problems for equations (1) and (1′) under conditions (8), (9), (9′).

* See also ([5]), p. 262.

** In ([10]) the case $m=1$ is considered; however, the results can be generalized to the case of any $m>0$ (cf. ([11])).

Thus, for example, with the aid of a result of Yu. Devingtal' ([11]), the existence and uniqueness of the solution of the Frankl problem for equation (1) (with the gluing condition (12)) is established, under the additional restriction on $\alpha$

$$(1-\gamma)/3<\alpha<1. \tag{21}$$

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*Note: Figure translations are in progress. See original paper for figures.*