ON LOGIC OF HIGHER LEVELS

1966

SovietRxiv

Abstract

Full Text

UDC 519.40

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(Presented by Academician P. S. Novikov, 19 II 1966)

In this note the results from ([1]), concerning logic of higher levels, are strengthened. Speaking of the models of one formula or another, we shall mean its standard models.

In ([1]) the following was proved.

Theorem I. *If a class of infinite structures is defined by a set of formulas of the k-th level, definable by a formula of the n-th level, then it is definable by a formula of the* $[\max(n,k)+1]$*-st level.*

With the help of Theorem I the following theorem can be proved, strengthening Theorem V from ([1]).

Theorem 1. *For every natural n there exists a closed (i.e., containing no free variables) formula of the* $(n+1)$*-st level which is not equivalent to any set of formulas of lower levels*.*

By the spectrum of a formula we shall mean the class of cardinalities of its models. The least of the cardinalities belonging to the spectrum of a satisfiable formula σ will be denoted by $k(\sigma)$. A cardinal $\mathfrak{n}$ will be called definable if there exists a formula all of whose models have cardinality $\mathfrak{n}$. Obviously, the cardinal $\mathfrak{n}$ is definable if $\mathfrak{n} = k(\sigma)$ for some formula σ.

It can be shown that the least of the cardinals not definable by any set of closed formulas of the n-th level is defined by a formula of the $(n+1)$-st level. This proves Theorem 1.

From Theorem I and from the fact that the notion of satisfiability for formulas of the n-th level is defined by a recursive set of formulas of the n-th level, there follows the well-known result of Tarski ([2]): the semantics of the language of the n-th level is expressible in the language of the $(n+1)$-st level.

Let F be the set of formulas of the n-th level (from $\mathfrak{S}$) that have no free variables other than object variables. Denote by E_n the formula in F with Gödel number n, and by $E_n(n)$ the formula obtained from E_n by replacing all occurrences of free variables by n (more precisely, by the term Δ_n, interpreted as the natural number n). By $D(n)$ denote the Gödel number of the formula $E_n(n)$. If, however, n is not the Gödel number of any formula in F, we put

$D(n) = 0$. Obviously, D is a recursive function and, hence, there exists an elementary formula Φ such that the truth of $\Phi(n, v)$ is equivalent to $v = D(n)$.

The proof of the following theorem, established in ([2]), is almost a word-for-word repetition of Theorem 1, Ch. II, from ([3]).

Theorem 2. *The set V of Gödel numbers of true formulas from F is not definable by a formula of the n-th level.*

Suppose, contrary to the assertion, that there exists a formula Ψ of the n-th level such that $n \in V$ is equivalent to the truth of $\Psi(n)$. Let m be the Gödel number of the formula (of the n-th level) $(\forall v)(\Phi(u, v) \to \neg\Psi(v))$. Hence,

$$E_m(m) = (\forall v)(\Phi(m, v) \to \neg\Psi(v)).$$

If $E_m(m)$ is true, then $\neg\Psi(D(m))$ is true. If $E_m(m)$ is false, then the number $E_m(m)$ does not belong to V, and hence $\neg\Psi(D(m))$ is true. Thus, $\neg\Psi(D(m))$ is true. But

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* Two sets of formulas are called equivalent if every model of one of them is a model of the other.

then the formula $E_m(m)$ is true. Hence, $D(m) \in V$, and therefore $\Psi(D(m))$ is true. We have arrived at a contradiction.

In ([1]) the following was proved (without using the axiom of choice).

Theorem. There exists an algorithm which, for every formula (of any finite order) σ, makes it possible to construct a formula $\Phi(\sigma)$ of second order such that the satisfiability (truth, categoricity) of σ is equivalent to the satisfiability (truth, categoricity) of $\Phi(\sigma)$.

Using this theorem and Theorem 2, it is not difficult to show that for every n there exists a second-order formula Φ such that the set of numbers of all formulas Ψ for which $\Phi \to \Psi$ is true is not definable by a formula of n-th order.

Hence one derives (without using the axiom of choice)

Theorem 3. The set of Gödel numbers of true formulas of the extended predicate calculus is not definable in arithmetic.

In ([1]) this theorem was proved with the aid of the axiom of choice. The use of the generalized continuum hypothesis proves

Theorem 4. For every cardinal $\mathfrak{n}$, the set $F_{\mathfrak{n}}$ of Gödel numbers of formulas of the extended predicate calculus that are true on sets of cardinality $\mathfrak{n}$ is definable if and only if $\mathfrak{n} < \aleph_\omega$. Moreover, for every natural n, the set $F_{\aleph_n}$ is definable by a formula of $(n + 1)$-st order.

At the same time it is easily proved that the set of numbers of all closed formulas of finite orders that are true on all finite sets is the complement of a recursively enumerable set.

It follows from this, in particular, that for the decidability of the theory of the class K_{fin}, consisting of the finite models of a class K defined by a single axiom, it is necessary and sufficient that this theory be recursively axiomatizable. (It also follows from this that if the theory of a class K defined by an elementary axiom is undecidable, then the theory of the class K_∞, consisting of the infinite models from K, is also undecidable.)

We shall call an ordinal α definable if there exists a formula of finite order that is true only on sets well ordered in type α. If every ordinal $\beta \leqslant \alpha$ is definable, then α will be called a strictly definable ordinal.

Analyzing the proof of Theorem II (contained in ([1])), one may observe that the following stronger statement is valid.

Theorem 5. For every strictly definable ordinal α there exists an algorithm which, from every formula σ of α-th order, constructs a formula $\Phi(\sigma)$ of second order such that the satisfiability (truth, categoricity) of σ is equivalent to the satisfiability (truth, categoricity) of $\Phi(\sigma)$, and $k(\sigma) < k(\Phi(\sigma))$ under the assumption of the axiom of choice.

Let Φ be the set of all satisfiable formulas of second order, and let k_0 be the least upper bound of the set $\{k(\varphi)\}_{\varphi\in\Phi}$. From Theorem 5 it follows, under the assumption of the axiom of choice, that for every strictly definable ordinal α and every formula σ of α-th order one has $k(\sigma) < k_0$.

Let Ψ be the set of formulas of second order satisfiable on infinite domains. For every formula φ from Ψ denote by φ^* the formula

$$(\exists X)(\widetilde{\varphi} \wedge \rho),$$

where $\widetilde{\varphi}$ is obtained from φ by relativizing the individual variables to X, and ρ asserts that X is infinite. Suppose that the set of Gödel numbers of formulas from Ψ is definable by a formula of α-th order for some strictly definable ordinal α. Then the set of Gödel numbers of formulas from the set $\{\varphi^*\}_{\varphi\in\Psi}$ is also definable by a formula of α-th order. The class defined by the latter set is defined by the formula σ^* of $(\alpha+1)$-st order. This set has no models of cardinalities less than k_0.

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* See ([1]), Theorem III.

On the other hand, it must be that $k(\sigma) < k_0$. The contradiction obtained proves the validity (under the assumption of the axiom of choice) of the following theorem, which strengthens Theorem 3.

Theorem 6. *For no strictly definable ordinal α is the set of Gödel numbers of true formulas of the extended predicate calculus definable by any formula of level α.*

Let us denote by A the set of closed formulas of the second level having the form $(\exists X)(Y)\mathfrak{A}(X, Y, x_1, \dots, x_n)$, where X is a four-place and Y a one-place predicate variable, and $\mathfrak{A}$ is a formula containing no quantifiers over predicate variables. From the fact that for every formula σ of the second level one can effectively construct a formula of the form A, whose truth is equivalent to the truth of σ, there follows the stronger assertion: for no strictly definable ordinal α is the set of Gödel numbers (for any Gödel numbering) of true formulas of the form A definable in arithmetic of level α.

We shall say that a class of cardinals C is predicatively defined by a sentence σ if it consists of cardinals $\aleph_\alpha$ such that well-ordered sets of ordinals α satisfy σ. If a class C is predicatively defined by a sentence of the n-th level, then it is the spectrum of some sentence of the n-th level. At the same time, with the help of the theorems given above, for Gödel set theory Σ^* one can prove that there exists a sentence of the second level whose spectrum is a set not predicatively definable by any sentence of level β, whatever the strictly definable ordinal β may be.

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Received
2 II 1966

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Note: Figure translations are in progress. See original paper for figures.