ON THE APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF FUNCTIONS BY FOURIER–JACOBI SUMS

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Let $\{\hat{P}_k^{(\alpha,\beta)}(x)\}_0^\infty$ be a system of Jacobi polynomials orthonormal on the interval $-1 \le x \le 1$ with weight $n(x) = (1-x)^\alpha(1+x)^\beta$ $(\alpha, \beta > -1)$, and let $R_n^{(\alpha,\beta)}(f,x) = f(x) - S_n^{(\alpha,\beta)}(f,x)$ be the n-th remainder of the Fourier–Jacobi series of a function $f(x)$ belonging to the class $W^{(r)}H^{(\gamma)}$ of all functions whose r-th derivative $(r \ge 0)$ on the interval $-1 \le x \le 1$ satisfies a Lipschitz condition of order γ $(0 < \gamma \le 1)$ with constant equal to one.

In the present note, for α and $\beta \ge 0$, an estimate is obtained for the quantity

$$\sup_{f\in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f,x)\|_C,$$

where $\|u(x)\|_C = \max_{-1\le x\le 1} |u(x)|$. In addition, for the case of ultraspherical polynomials $(\alpha = \beta = \lambda - \frac{1}{2})$ it is proved that the estimate obtained is exact in the sense of order.

In what follows $q = \max(\alpha, \beta)$, and all constants $C_1, \ldots$ depend neither on x, nor on n, nor on the function $f \in W^{(r)}H^{(\gamma)}$.

Theorem 1. Let $r \ge 0$, $0 < \gamma \le 1$, $r + \gamma > \frac{1}{2}$. If $\alpha, \beta > 0$, $q \ge \frac{1}{2}$, then the estimate

$$\sup_{f\in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f,x)\|_C \le \frac{C_1}{(n+1)^{r+\gamma-q-\frac{1}{2}}}, \tag{1}$$

is valid, while if $\alpha, \beta > 0$, $q < \frac{1}{2}$, or if $\alpha\beta = 0$ $(\alpha \ge 0,\ \beta \ge 0)$, then

$$\sup_{f\in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f,x)\|_C \le \frac{C_2 \ln(n+2)}{(n+1)^{r+\gamma-q-\frac{1}{2}}}. \tag{2}$$

Theorem 2. Let $a > -\frac{1}{2}$, $r \ge 0$, $0 < \gamma \le 1$. Then there exists a constant $C_3 > 0$ such that

$$\sup_{f\in W^{(r)}H^{(\gamma)}} |R_n^{(\alpha,\alpha)}(f,1)| \geq \frac{C_3}{(n+1)^{r+\gamma-\alpha-\frac{1}{2}}}. \tag{3}$$

The proof of Theorem 1 is carried out with the aid of the strengthened Jackson theorem (see ([1]), p. 276), by virtue of which, for any $n > r$, for every function $f(x) \in W^{(r)}H^{(\gamma)}$ one can construct an algebraic polynomial $Q_n(x)$ of degree $\leq n$ such that

$$|f(x) - Q_n(x)| \leq \frac{C_4}{n^{r+\gamma}}\left[(\sqrt{1-x^2})^{r+\gamma} + \frac{1}{n^{r+\gamma}}\right] \qquad (-1 \leq x \leq 1).$$

It follows from this that

$$|R_n^{(\alpha,\beta)}(f,x)| \leq |f(x) - Q_n(x)| + \int_{-1}^{1} |f(t) - Q_n(t)|\,|K_n^{(\alpha,\beta)}(x,t)|\,n(t)\,dt \leq$$

$$\leq \frac{C_5}{(n+1)^{r+\gamma}} + \frac{C_6}{(n+1)^{r+\gamma}} \int_{-1}^{1} (1-t)^{r/2+\gamma/2+\alpha}(1+t)^{r/2+\gamma/2+\beta}|K_n^{(\alpha,\beta)}(x,t)|\,dt +$$

$$+\frac{C_7}{(n+1)^{2(r+\gamma)}} \int_{-1}^{1} \left|K_n^{(\alpha,\beta)}(x,t)\right| n(t)\,dt =$$

$$= \frac{C_5}{(n+1)^{r+\gamma}} + \frac{C_6}{(n+1)^{r+\gamma}} I_1 + \frac{C_7}{(n+1)^{2(r+\gamma)}} I_2, \tag{4}$$

where

$$K_n^{(\alpha,\beta)}(x,t) = \sum_{k=0}^{n} \hat{P}_k^{(\alpha,\beta)}(x)\hat{P}_k^{(\alpha,\beta)}(t).$$

For the integral I_2 we have

$$I_2 \leq \left\{\int_{-1}^{1} \left|K_n^{(\alpha,\beta)}(x,t)\right|^2 n(t)\,dt\right\}^{1/2} \left\{\int_{-1}^{1} n(t)\,dt\right\}^{1/2} = \tag{5}$$

$$= \left\{\sum_{k=0}^{n} \left|\hat{P}_k^{(\alpha,\beta)}(x)\right|^2\right\}^{1/2} \left\{\int_{-1}^{1} n(t)\,dt\right\}^{1/2} \leq C_8 \left\{\sum_{k=0}^{n} (k+1)^{2q+1}\right\}^{1/2} \leq C_9 (n+1)^{q+1}.$$

From estimate (5) it follows that the last term in (4) is $O((n+1)^{q+1-2r-2\gamma})$ uniformly in x and f.

The estimate of the integral I_1 in the case $\alpha, \beta > 0,\ q \geq 1/2$ is carried out with the help of the formula

$$P_{n-1}^{(\alpha,\beta)}(x) = \frac{1}{(1-x)^{\alpha/2+1}(1+x)^{\beta/2+1}} \times$$

$$\times \left\{ \int_{-1}^{x} (1-t)^{\alpha/2}(1+t)^{\beta/2} \left[\frac{\alpha}{2} - \frac{\beta}{2} + \left(\frac{\alpha}{2} + \frac{\beta}{2} - 2 \right) t \right] P_{n-1}^{(\alpha,\beta)}(t)\, dt \right.$$

$$\left. -2n \int_{-1}^{x} (1-t)^{\alpha/2}(1+t)^{\beta/2} P_n^{(\alpha-1,\beta-1)}(t)\, dt \right\} \quad (\alpha, \beta > 0), \tag{6}$$

which follows from the differential equation for Jacobi polynomials. From this formula it follows that for arbitrary α and $\beta > 0$ for which $q \geq 1/2$, and for $-1 < t < 1$, the inequality

$$\left| \frac{\hat{P}_n^{(\alpha,\beta)}(x) - \hat{P}_n^{(\alpha,\beta)}(t)}{x - t} \right| \leq \frac{C_{10}(n+1)^{q+1/2}}{(1-t)^{\alpha/2+1}(1+t)^{\beta/2+1}} \quad (-1 \leq x \leq 1), \tag{7}$$

holds, where the constant C_{10} depends neither on x, nor on t, nor on n. Using the Christoffel-Darboux formula ((2), p. 83) and inequality (7), for α and $\beta > 0,\ q \geq 1/2$ we obtain

$$I_1 \leq C_{11}(n+1)^{q+1/2} \times$$

$$\times \int_{-1}^{1} (1-t)^{(r+\gamma+\alpha-2)/2}(1+t)^{(r+\gamma+\beta-2)/2} \left[\left| \hat{P}_n^{(\alpha,\beta)}(t) \right| + \left| \hat{P}_{n+1}^{(\alpha,\beta)}(t) \right| \right] dt. \tag{8}$$

But the integral in inequality (8) is bounded as $n \to \infty$ (see (2), p. 180). From inequalities (4), (5), and (8) the validity of estimate (1) follows.

In the cases where $\alpha, \beta > 0,\ q < 1/2$, or where $\alpha\beta = 0$ ($\alpha \geq 0,\ \beta \geq 0$), the integral I_1 is estimated with the help of a device applied by P. K. Suetin (5) to Legendre polynomials ($\alpha = \beta = 0$), and of an inequality proved by S. N. Bernstein (3):

$$\left| \hat{P}_n^{(\alpha,\beta)}(x) \right| (1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4} \leq C_{12} \quad (-1 \leq x \leq 1).$$

In proving Theorem 2 it is enough to restrict oneself to the consideration of functions $f(x) \in W^{(r)}H^{(\gamma)}$, for which $f(1) = f'(1) = \dots = f^{(r)}(1) = 0$.

We have, for any $\lambda > 0$ and $0 < \theta < \pi$,

$$(\sin\theta)^{2\lambda-1} P_n^{(\lambda)}(\cos\theta) = \frac{2^{2-2\lambda}}{\Gamma(\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} \sin(n+2\nu+1)\theta, \quad (9)$$

$$f_{0n}^{(\lambda)} = 1; \qquad f_{\nu n}^{(\lambda)} = \frac{(1-\lambda)(2-\lambda)\cdots(\nu-\lambda)}{\nu!} \times$$

$$\times \frac{(n+1)(n+2)\cdots(n+\nu)}{(n+\lambda+1)(n+\lambda+2)\cdots(n+\lambda+\nu)}, \qquad \nu = 1, 2, \ldots$$

In the case of nonintegral λ, this formula is given by G. Szegő ([2]). Using the uniform convergence in λ of the series in (9), one can prove formula (9) for natural $\lambda = k$, with $f_{\nu n}^{(k)} = 0$ if $\nu \geq k$. With the aid of representation (9), the expression for the remainder $R_n^{(\lambda-1/2,\lambda-1/2)}(f,1) = R_n^{(\lambda)}$, for arbitrary $\lambda > 0$, is brought to the form

$$R_n^{(\lambda)} = a_n^{(\lambda)} \left\{ \int_0^{2\pi} f(\cos\psi) \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} \left[\cos(n+2\nu+1)\psi + \right.\right.$$

$$\left. + \left(1 - \frac{\lambda}{n+\nu+1+\lambda}\right) \cos(n+2\nu+2)\psi \right] d\psi -$$

$$\left. - \int_0^{2\pi} f(\cos\psi) \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} \frac{2\lambda}{n+\nu+1+\lambda} \frac{\sin(n+2\nu+\frac{3}{2})\psi}{2\sin\psi/2} \, d\psi \right\}, \qquad (10)$$

where

$$a_n^{(\lambda)} = \Gamma(n+2\lambda+1) \left[2^{2\lambda}\sqrt{\pi}\,\Gamma(\lambda+\tfrac{1}{2})\Gamma(n+\lambda+1)\right]^{-1}.$$

The application of r-fold integration by parts and the Abel transformation in the first integral in (10), as well as the application of Jackson' s estimate ([4]) for the n-th remainder of the Fourier series of a 2π-periodic differentiable function in estimating the second integral in (10), makes it possible to bring expression (10) to the form

$$R_n^{(\lambda)} = a_n^{(\lambda)} \left\{ \frac{(-1)^r 2^{\lambda+1}}{(n+1)^r} \int_0^{\pi} \cos\left[\left(n + \frac{1}{2} + \lambda\right)\psi + \frac{(r+1-\lambda)\pi}{2}\right] \times \right.$$

$$\left. \times \cos\frac{\psi}{2} \sin^{r+\lambda-1}\psi \, f^{(r)}(\cos\psi) \, d\psi + I_3 \right\}, \qquad (11)$$

where

$$I_3 = O\left(\frac{\ln(n+2)}{(n+1)^{r+\gamma+1}}\right), \qquad \text{if } \lambda \geq 1;$$

$$I_3 = O\left(\frac{\ln(n+2)}{(n+1)^{r+\gamma+\varepsilon}}\right), \qquad \text{if } 0 < \varepsilon < \lambda < 1.$$

From (11) and from the fact that

$$\sup_{\varphi(x)\in H^{(\gamma)}} \left| \int_0^\pi \cos\left[\left(n+\frac{1}{2}+\lambda\right)\psi + \frac{(r+1-\lambda)\pi}{2}\right] \cos\frac{\psi}{2} \sin^{r+\lambda-1}\psi\, \varphi(\cos\psi)\, d\psi \right|$$

has order $n^{-\gamma}$, the validity of Theorem 2 follows. The indicated order is attained on the function $v_n(x) \in H^{(\gamma)}$ $(-1 \leq x \leq 1)$, defined by the formula

$$v_n(x) = \frac{1}{2} u_n(x) \operatorname{sign}\left\{\cos\left[\left(n+\frac{1}{2}+\lambda\right)\arccos x + \frac{(r+1-\lambda)\pi}{2}\right]\right\},$$

where $u_n(x) = (x-a)^\gamma$ for $x \in [a, (a+b)/2]$ and $u_n(x) = (b-x)^\gamma$ for $x \in [(a+b)/2, b]$, while $[a, b]$ is any one of the parts into which the segment

$[-1, 1]$ is divided by the zeros of the function

$$\cos\left[\left(n+\frac{1}{2}+\lambda\right)\arccos x + \frac{(r+1-\lambda)\pi}{2}\right].$$

Let us note that in the case when $\alpha = \beta$ is an integer, inequality (1) follows from the estimate of the Lebesgue function of the Fourier–Jacobi series obtained by I. K. Daugavet [6].

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Note: Figure translations are in progress. See original paper for figures.