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Abstract

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MATHEMATICS

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A CRITERION FOR THE DISCRETENESS OF THE SPECTRUM OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS WITH SLOWLY VARYING COEFFICIENTS

(Presented by Academician I. M. Vinogradov on VII 24, 1965)

We consider the self-adjoint operator in $L_2(-\infty, +\infty)$

$$D_0 y = i\Lambda_0 y' + Q_0(x)y(x) = i\begin{pmatrix}\lambda_1, & 0\\ 0, & \lambda_2\end{pmatrix} y' + \begin{pmatrix}p, & q\\ \bar{q}, & r\end{pmatrix} y, \tag{1}$$

where the real numbers $\lambda_1 \cdot \lambda_2 \neq 0$, and the matrix-function $Q_0(x)$ is continuous everywhere on the axis and Hermitian in the unitary space E_2. If $\lambda_1 = -\lambda_2 = 1$ and $p(x) \equiv r(x)$, then, in essence, we are dealing with the operator

$$T_0 y = I_0 y' + A_0(x)y(x) = \begin{pmatrix}0, & 1\\ -1, & 0\end{pmatrix} y' + \begin{pmatrix}a, & b\\ b, & c\end{pmatrix} y \tag{2}$$

in the Euclidean space R_2, since $T_0 = UD_0U^{-1}$ for $U = \frac{1}{\sqrt{2}}\begin{pmatrix}-i, & i\\ 1, & 1\end{pmatrix}$. The operator T_0 was studied in a number of papers by E. C. Titchmarsh in connection with the relativistic Dirac equation.

In note ([2]) certain sufficient conditions were proposed for discreteness of the spectrum of the operator D_0. The main content of the present note is the proof of a necessary and sufficient condition for discreteness of the spectrum, valid in the class of operators with slowly varying coefficients.

Theorem. 1) *Let the matrix $Q_0(x)$ vary slowly, i.e.*

$$\|Q_0(x) - Q_0(t)\| \leq \text{const}* \tag{3}$$

for all $|x - t| \leq 1$. Then the spectrum of the operator is discrete if and only if

$$\lim_{|x|\to\infty}\Big[\,|q(x)|\sqrt{-4\lambda_1\lambda_2}-|\lambda_2 p(x)-\lambda_1 r(x)|\,\Big]=+\infty. \tag{4}$$

2) *If the matrix $Q_0(x)$ varies slowly, then the continuous spectrum of the operator D_0 is bounded above (it may also be empty) if and only if*

$$\lim_{\lambda\to+\infty}\ \lim_{|x|\to\infty}\Big[\,|q(x)|\sqrt{-4\lambda_1\lambda_2}-|(\lambda_2 p-\lambda_1 r)+\lambda(\lambda_1-\lambda_2)|\,\Big]=+\infty. \tag{5}$$

3) *If the matrix $Q_0(x)$ varies slowly, then the continuous spectrum of the operator D_0 is bounded below (it may also be empty) if and only if*

$$\lim_{\lambda\to-\infty}\ \lim_{|x|\to\infty}\Big[\,|q(x)|\sqrt{-4\lambda_1\lambda_2}-|(\lambda_2 p-\lambda_1 r)+\lambda(\lambda_1-\lambda_2)|\,\Big]=+\infty. \tag{6}$$

Corollary. *If $\lambda_1=-\lambda_2=1$ and $p(x)\equiv r(x)$, then conditions (4), (5), (6) admit a reformulation in terms of the eigenvalues of the matrix $Q_0(x)$ (and hence also of the matrix $A_0(x)$):*

* $\|B\|$, $\mu[B]$, and $\nu[B]$ will denote, respectively, the Euclidean norm, the smallest and the largest eigenvalues o

1′) the spectrum is discrete if and only if

$$\lim_{|x|\to\infty}\nu[A_0(x)]=+\infty,\qquad \lim_{|x|\to\infty}\mu[A_0(x)]=-\infty; \tag{7}$$

2′) the continuous spectrum is bounded above if and only if

$$\lim_{|x|\to\infty}\nu[A_0(x)]=+\infty\quad\text{and the function }\mu[A_0(x)]\text{ is bounded above;} \tag{8}$$

3′) the continuous spectrum is bounded below if and only if

$$\lim_{|x|\to\infty}\mu[A_0(x)]=-\infty\quad\text{and the function }\nu[A_0(x)]\text{ is bounded below.} \tag{9}$$

Thus, in this case the spectral alternative is valid (cf. ([1]), p. 174): the continuous part of the spectrum of the operator D_0 either is absent, or else is unbounded at least on one side.

Proceeding to the proof of the theorem, consider, as in ([2]), the more general operator

$$Dy=i\Lambda y'+Q(x)y(x), \tag{10}$$

where the principal part contains a constant real diagonal invertible matrix Λ, and the matrix function $Q(x)$ is continuous and Hermitian in the unitary E_k ($k \geqslant 2$).

Lemma 1 (splitting principle; cf. ([1]), §§ 1, 2). *The distance from a real point λ to the continuous spectrum of the operator (10) is equal to*

$$\lim_{r\to+\infty}\left(\inf_{y\in K(r,\infty)}\frac{\|Dy-\lambda y\|}{\|y\|}\right), \tag{11}$$

where $K(r,\infty)$ denotes the class of finite piecewise-smooth vector functions, the compact support of each of which is situated outside the interval $(-r,+r)$.

Corollary 1. *In order that the spectrum of the operator D be discrete, it is necessary and sufficient that*

$$\lim_{r\to+\infty}\left(\inf_{y\in K(r,\infty)}\frac{\|Dy\|}{\|y\|}\right)=+\infty. \tag{12}$$

Corollary 2. *In order that the continuous part of the spectrum of the operator D be bounded above, it is necessary and sufficient that*

$$\lim_{\lambda\to+\infty}\lim_{r\to+\infty}\left(\inf_{y\in K(r,\infty)}\frac{\|Dy-\lambda y\|}{\|y\|}\right)=+\infty. \tag{13}$$

The condition for boundedness of the continuous spectrum below and for its two-sided boundedness is equivalent, respectively, to the requirements that the expression (11) tend to infinity as $\lambda\to-\infty$ and as $|\lambda|\to\infty$.

Lemma 2 (splitting principle; cf. ([1]), p. 66). *Put*

$$d(\lambda;\Delta)=d(\lambda;a,b)=\inf_{y\in N(\Delta)}\frac{\|Dy-\lambda y\|^2}{\|y\|^2},$$

where the class $N(\Delta)=N(a,b)$ consists of finite piecewise-smooth vector functions vanishing outside the interval $\Delta=[a,b]$. For any number h $(0<h<b-a\leqslant\infty)$ there exists in Δ an interval Δ' of length h such that

$$d(\lambda;\Delta)\leqslant d(\lambda;\Delta')\leqslant d(\lambda;\Delta)+\frac{32}{h^2}\|\Lambda\|^2. \tag{14}$$

Corollary 1′. The spectrum of the operator D is discrete if and only if

$$\lim_{|x|\to\infty} d(0;x,x+1)=+\infty. \tag{15}$$

Corollary 2′. The continuous part of the spectrum of the operator D is bounded above if and only if

$$\lim_{\lambda\to+\infty}\lim_{|x|\to\infty} d(\lambda; x, x+1) = +\infty. \tag{16}$$

The conditions for boundedness of the continuous spectrum below and for its two-sided boundedness coincide with (16) for $\lambda\to-\infty$ and $|\lambda|\to\infty$, respectively.

Suppose now that the matrix Q varies slowly. Then $d(0;\alpha,\alpha+1) = m(\alpha)+O(1)$, where

$$m(\alpha) = \inf_{y\in N(\alpha,\alpha+1)}\left[\int_{\alpha}^{\alpha+1}\left|i\Lambda y'(t)+Q(\alpha)y(t)\right|^2 dt\Big/\int_{\alpha}^{\alpha+1}|y(t)|^2\,dt\right].$$

Consequently, the spectrum is discrete if $\lim_{|\alpha|\to\infty} m(\alpha) = +\infty$. But, on the other hand, by virtue of inequality (14), $m(\alpha) = n(\alpha)+O(1)$, where

$$n(\alpha) = \inf_{y\in N(-\infty,+\infty)}\left[\int_{-\infty}^{+\infty}\left|i\Lambda y'(t)+Q(\alpha)y(t)\right|^2 dt\Big/\int_{-\infty}^{+\infty}|y(t)|^2 dt\right].$$

Thus the spectrum is discrete if and only if $\lim_{|\alpha|\to\infty} n(\alpha) = +\infty$. To compute $n(\alpha)$, we apply the Fourier transform:

$$\begin{aligned} n(\alpha) &= \inf_z\left[\int_{-\infty}^{+\infty}\left|Q(\alpha)z(t)-t\Lambda z(t)\right|^2 dt\Big/\int_{-\infty}^{+\infty}|z(t)|^2 dt\right] \\ &= \inf_{t\in(-\infty,+\infty)}\mu\left[(Q(\alpha)-t\Lambda)^2\right] \\ &= \inf_t\min_{|\xi|=1}\left[(\Lambda^2\xi,\xi)t^2-2(\operatorname{Re}\Lambda Q\xi,\xi)t+(Q^2(\alpha)\xi,\xi)\right]. \end{aligned}$$

Thus we finally obtain: the spectrum of the operator D is discrete if and only if

$$\lim_{|\alpha|\to\infty}\left[\min_{|\xi|=1} X(\alpha;\xi)\right] = +\infty, \tag{17}$$

where

$$X(\alpha;\xi) = (\Lambda^2\xi,\xi)(Q^2(\alpha)\xi,\xi)-\left[\operatorname{Re}(\Lambda Q\xi,\xi)\right]^2.$$

It is not difficult to show that for $k=2$ condition (17) is equivalent to condition (4) of the theorem.

Remark. The lemmas and condition (17) remain valid if in the principal part of the operator D there stands an arbitrary (not necessarily diagonal) constant Hermitian matrix Λ.

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References Cited

1. I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, 1963.

2. V. V. Martynov, DAN, 165, No. 5 (1965).

Note: Figure translations are in progress. See original paper for figures.