ON FUNCTIONS OF SPECIAL OPERATORS IN THE THEORY OF ELASTIC-HEREDITARY MEDIA

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Abstract

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ON FUNCTIONS OF SPECIAL OPERATORS IN THE THEORY OF ELASTIC-HEREDITARY MEDIA

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The effective solution of many complex problems in the theory of elastic-hereditary media on the basis of V. Volterra' s principle (^{1}), by bringing in the corresponding solutions of elastic problems, when the latter are expressed through irrational and transcendental functions of elastic constants, can be carried out only by deciphering the corresponding functions of integral operators.

Consider a function of the coordinates $x(x_1, x_2, x_3)$ and of integral operators $\mathcal{E}_\alpha^*(-\beta_i)$ with fractional-exponential kernels of the type of Yu. N. Rabotnov (^{1}), with rheological parameters $\beta_i > 0$ $(i = 1, 2, \dots, n)$, represented in the form of the expansion

$$f[x; \mathcal{E}_\alpha^*(-\beta_1), \dots, \mathcal{E}_\alpha^*(-\beta_n)] =$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n}(x) D_m \sum_{i=1}^{n} \left[\prod_{k=1}^{n} (\beta_i - \beta_k) \right]^{-1} \mathcal{E}_\alpha^*(-\beta_i), \tag{1}$$

where the differential operator

$$D_m = \frac{(-1)^{m_1+\dots+m_n-1}}{\prod_{i=1}^{n}(m_i-1)!} \frac{\partial^{m_1+\dots+m_n-n}}{\partial\beta_1^{m_1-1} \dots \partial\beta_n^{m_n-1}}. \tag{2}$$

Expansion (3) is carried out in accordance with the properties of the operators $\mathcal{E}_\alpha^*$, established by M. I. Rozovskii (^{2}).

For uniform convergence of series (1) in any finite domain of variation of the arguments, it is sufficient that the corresponding majorizing power series for $f(\xi_1, \dots, \xi_n)$ converge at least for arbitrarily small values of ξ_i. The validity of the latter is confirmed by Ya. V. Bykov' s theorem (^{3}) on the convergence of composition series of permutable functions.

When the function of operators under consideration acts on some function of time, then

$$\mathcal{E}_{\alpha}^{*}(-\beta_i)\varphi(t) = \int_0^t \sum_{\nu=0}^{\infty} \frac{(-\beta_i)^{\nu}(t-\tau)^{\nu(1+\alpha)+\alpha}}{\Gamma[(\nu+1)(1+\alpha)]}\varphi(\tau)\,d\tau, \qquad -1 < \alpha \leq 0. \tag{3}$$

Thus, representations (1)–(3) make it possible in principle to solve the problem posed. However, such a solution is of little use for qualitative analysis.

Consider the characteristic cases when $\varphi(t) \approx \varphi(0)$ for small t, and when $\varphi(t) \approx \varphi(\infty)$ for sufficiently large t, where $\varphi(\infty)$ is the established finite value of $\varphi(t)$ as $t \to \infty$.

In the first case the approximation holds

$$\mathcal{E}_{\alpha}^{*}(-\beta_i)\varphi(t) \approx \frac{q\varphi(t)}{1+q\beta_i}, \tag{4}$$

where $q\beta_i < 1$, $q = t^{1+\alpha}[\Gamma(2+\alpha)]^{-1}$, obtained on the basis of the approximation given in (^{4}), with error

$$\Delta < Mq^2\beta_i^2[\Gamma(2\alpha+3)]^{-1}, \qquad M = \max\varphi(t).$$

From representation (1), taking (4) into account, we obtain

$$f\left[x; \mathfrak{D}_{\alpha}^{*}(-\beta_1), \dots, \mathfrak{D}_{\alpha}^{*}(-\beta)\right] \approx$$

$$\approx \sum_{m_1,\dots,m_n=0}^{\infty} \alpha_{m_1,\dots,m_n} D_m \sum_{i=1}^{n} \frac{q\varphi(t)}{(1+q\beta_i)\prod\limits_{k=1}^{n}(\beta_i-\beta_k)}. \tag{5}$$

After summing the series (5), we shall have

$$f\left[x; \mathfrak{D}_{\alpha}^{*}(-\beta_1), \dots, \mathfrak{D}_{\alpha}^{*}(-\beta_n)\right]\varphi(t) \approx f\left(x; \frac{q}{1+q\beta_1}, \dots, \frac{q}{1+q\beta_n}\right)\varphi(t), \tag{6}$$

where q is the function of time t indicated above.

In the second case, when $\varphi(t) \approx \varphi(\infty)$ and $\beta_i t^{1+\alpha} > 1$, it is expedient to use the approximation

$$\mathfrak{D}_\alpha^*(-\beta_i)\varphi(t) \approx \frac{\varphi(t)}{\beta_i}\left[1 - \sum_{\nu=1}^{N}(-1)^{\nu-1}\frac{\beta_i^{-\nu}t^{-\nu(1+\alpha)}}{\Gamma(1-\nu-\alpha\nu)}\right], \tag{7}$$

obtained on the basis of the expression of the operator $\mathfrak{D}_\alpha^*$ through the Mittag-Leffler function (^{4}) and the corresponding asymptotics (^{5}).

From representation (1), taking (8) into account, we obtain

$$f\left[x; \mathfrak{D}_\alpha^*(-\beta_1), \dots, \mathfrak{D}_\alpha^*(-\beta_n)\right]\varphi(t) \approx$$

$$\approx \left\{ \sum_{m_1,\dots,m_n=0}^{\infty} \alpha_{m_1,\dots,m_n} D_m \sum_{i=1}^{n} \frac{1}{\beta_i \prod_{k=1}^{n}(\beta_i - \beta_k)} \right.$$

$$\left. - \sum_{m_1,\dots,m_n=0}^{\infty} \alpha_{m_1,\dots,m_n} D_m \sum_{i=1}^{n} \frac{1}{\beta_i \prod_{k=1}^{n}(\beta_i - \beta_k)} \sum_{\nu=1}^{N}(-1)^{\nu-1}\frac{\beta_i^{-\nu}t^{-\nu(1+\alpha)}}{\Gamma(1-\nu-\alpha\nu)} \right\} \varphi(t). \tag{8}$$

The first series in representation (8) is reduced to the form

$$\sum_{m_1,\dots,m_n=0}^{\infty} \alpha_{m_1,\dots,m_n} \frac{1}{\prod_{i=1}^{n} \beta_i^{m_i}} = f\left(x; \frac{2}{\beta_1}, \dots, \frac{1}{\beta_n}\right). \tag{9}$$

The summation of the second series in representation (8) is carried out by invoking the formula

$$\sum_{\nu=1}^{N}\sum_{i=1}^{n} \frac{1}{\beta_i^{\nu+1}\prod_{k=1}^{n}(\beta_i - \beta_k)} = \sum_{\nu=1}^{N} \frac{(-1)^{n+\nu-1}}{\nu!}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{(\nu)} \frac{1}{\prod_{i=1}^{n}\beta_i}, \tag{10}$$

where

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{(\nu)}=\sum\frac{\nu!}{\alpha_1!\cdots\alpha_\nu!}\frac{\partial^{\alpha_1+\cdots+\alpha_\nu}}{\partial\beta_1^{\alpha_1}\cdots\partial\beta_\nu^{\alpha_\nu}}. \tag{11}$$

In expression (11), the summation extends over all possible groups of nonnegative integers α_i $(i=1,2,\dots,\nu)$. Then the expression for the second series in representation (8) is transformed into the form

$$\sum_{m_1,\dots,m_n=0}^{\infty}\alpha_{m_1,\dots,m_n}D_m\sum_{\nu=1}^{N}\frac{(-1)^n t^{-\nu(1+\alpha)}}{\nu!\,\Gamma(1-\nu-\alpha\nu)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{\nu}\frac{1}{\prod\limits_{i=1}^{n}\beta_i}=$$

$$=\sum_{\nu=1}^{N}\frac{t^{-\nu(1+\alpha)}}{\nu!\Gamma(1-\nu-\alpha\nu)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{(\nu)}f\left(x;\frac{1}{\beta_1},\dots,\frac{1}{\beta_n}\right). \tag{12}$$

Taking into account the representations (9) and (12), we finally obtain, for sufficiently large t,

$$f[x;\mathcal{E}_\alpha^*(-\beta_1),\dots,\mathcal{E}_\alpha^*(-\beta_n)]\varphi(t)\simeq\left[f\left(x;\frac{1}{\beta_1},\dots,\frac{1}{\beta_n}\right)+\sum_{\nu=1}^{N}\frac{t^{-\nu(1+\alpha)}}{\nu!\Gamma(1-\nu-\alpha\nu)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{(\nu)}f\left(x;\frac{1}{\beta_1},\dots,\frac{1}{\beta_n}\right)\right] \tag{13}$$

Here the error corresponding to (13) is estimated by using the estimate of the error (5) of approximation (8) as follows:

$$r_N(t)\le\frac{(-1)^{N+1}\Gamma[(1+\alpha)N+1+\alpha]\varphi(\infty)}{(N+2)!\pi t^{1+\alpha+N(1+\alpha)}}\left(\sum_{i=1}^{n}\frac{\partial}{\partial\beta_i}\right)^{(N+2)}f\left(x;\frac{1}{\beta_1},\dots,\frac{1}{\beta_n}\right).$$

The representations (6) and (13) of functions of operators make it possible to study effectively the solutions of complicated problems in the theory of elastic-hereditary bodies.

As an application, let us determine the stresses σ_r and σ_θ arising in a continuous elastic-hereditary disk with cylindrical anisotropy, compressed along the circumference by constant normal forces p_0. For this purpose we first form, according to V. Volterra' s principle, the operator expressions

$$\sigma_r=-p_0(r/b)^{\bar{k}-1},\qquad\sigma_\theta=-p_0\bar{k}(r/b)^{\bar{k}-1}, \tag{14}$$

where

$$\bar{k} = k\sqrt{1 + \chi_1 \mathcal{E}^*_\alpha(-\beta_1) + \chi_2 \mathcal{E}^*_\alpha(-\beta_2)},$$

obtained by applying the known formulas (6) corresponding to the corresponding elastic problem, in which the moduli E_r and E_θ are replaced by the operators

$$\bar{E}_r = E_{r0}[1 - \chi_r \mathcal{E}^*_\alpha(-\beta_r)],$$
$$\bar{E}_\theta = E_{\theta 0}[1 - \chi_\theta \mathcal{E}^*_\alpha(-\beta_\theta)], \qquad \beta_r > \chi_r, \ \beta_\theta > \chi_\theta,$$

with subsequent use of the basic properties of $\mathcal{E}^*_\alpha$-operators. Here

$$\chi_1 = (\beta_\theta \chi_\theta - \beta_r \chi_\theta)\lambda^{-1}, \qquad \chi_2 = (\chi_\theta \chi_r + \chi_r \beta_r - \chi_r \beta_\theta - \chi_r^2)\lambda^{-1},$$

$$\lambda = \beta_r - \beta_\theta - \chi_r, \qquad \beta_1 = \beta_\theta, \quad \beta_2 = \beta_r - \chi_r, \qquad k_0 = \sqrt{E_{\theta 0}/E_{r0}}.$$

According to representation (6), for the case $\beta_i t^{1+\alpha} < \Gamma(2+\alpha)$ the problem reduces to replacing, in (14), the operator $\bar{k}$ by its expression

$$\bar{k} = k_0 \left[1 + \sum_{i=1}^{2} \chi_i \left(1 - \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha) + \beta_i t^{1+\alpha}}\right)\right]^{1/2}. \tag{15}$$

In the case $\beta_i t^{1+\alpha} \gg 1$, according to (13), we shall have, for $N = 1$,

$$\sigma_r(t) = \sigma_r(\infty) + \frac{1}{t^{1+\alpha}\Gamma(-\alpha)} \left(\frac{\partial}{\partial \beta_\theta} + \frac{\partial}{\partial(\beta_r - \chi_r)}\right) \sigma_r(\infty). \tag{16}$$

The stress $\sigma_\theta(t)$ is obtained from (16) by replacing the index r by θ. The quantities $\sigma_r(\infty)$ and $\sigma_\theta(\infty)$ are determined by means of formulas (14) by replacing in them the operator $\bar{k}$ by its limiting value

$$k_\infty = k_0 \left[\frac{(\beta_\theta - \chi_\theta)\beta_r}{(\beta_r - \chi_r)\beta_\theta}\right]^{1/2}. \tag{17}$$

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Note: Figure translations are in progress. See original paper for figures.