ON CONDITIONS FOR THE SOLVABILITY OF THE CHAPLYGIN PROBLEM

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Abstract

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MATHEMATICS

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ON CONDITIONS FOR THE SOLVABILITY OF THE CHAPLYGIN PROBLEM

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Consider on the interval $[a, b]$ of the real axis t the equation

$$x' = F(t, x), \tag{1}$$

where the operator $w = F(t, x)$ maps $t \in [a, b]$ and $x \in X$ into $w \in X$; X is a KB-lineal ([1,2]); the operator $w = F(t, x)$ is continuous in the variables $t \in [a, b]$, $x \in X$; $C^1(a, b)$ is the space of continuously differentiable functions on the interval $[a, b]$ with values in X.

We shall say that on the interval $[a, b]$ the Chaplygin problem is solvable for equation (1) if, on the interval $[a, b]$, for any element $x_0 \in X$ and any function $z(t) \in C^1(a, b)$ satisfying the conditions $z(a) = x_0$, $z' > F(t, z)$, the inequality $z' > F(t, z)$ implies the inequality $z(t) \geq x(t)$, where $x(t) \in C^1(a, b)$ is the solution of (1) satisfying the condition $x(a) = x_0$.

Introduce in X a bilinear functional, equal to $x \cdot y$ $(x, y \in X)$, having the following properties: 1) from the inequality $x \cdot y \leq 0$ for every element $x \geq 0$ $(x \in X)$, with a fixed element $y \in X$, it follows that $y \leq 0$; 2) from $x \geq 0$, $y \leq 0$ $(x, y \in X)$, it follows that $x \cdot y \leq 0$.

Suppose that the operator $F(t, x)$ is differentiable with respect to x and that there exists an operator $[F'_x(t, x)]^*$, linear in $\xi \in X$, with values in X, satisfying the condition: for any $t \in [a, b]$ and any elements $x, g, \xi \in X$, $[F'_x(t, x)g] \cdot \xi = g \cdot [F'_x(t, x)]^* \xi$. Denote by $\psi(t)$ the solution of the problem

$$\psi' = -[F'_x(t, x)]^* \psi, \qquad \psi(\tau) = \psi_0 \quad (\psi_0 \in X), \tag{2}$$

where $x = x(t) \in C^1(a, b)$ is a solution of equation (1), $\psi_0 \in X$, and $a \leq \tau \leq b$.

Theorem 1. *Let, in equations (1) and (2), the operators $F(t, x)$ and $[F'_x(t, x)]^*$ satisfy the conditions: 1) the operator $F(t, x)$ has derivatives with respect to x up to second order inclusive, continuous in $t \in [a, b]$ and $x \in X$; 2) the operator $[F'_x(t, x)]^*$ has a derivative with respect to x, continuous in $t \in [a, b]$ and $x \in X$.*

Then, for the solvability on the interval $[a,b]$ of the Chaplygin problem for equation (1), it is necessary, and in the case when the operator $F(t,x)$ is linear in x also sufficient, that for any τ $(a<\tau\le b)$ and any $\psi_0<0$ the solution $\psi(t)$ of problem (2) satisfy on the interval $[a,\tau]$ the condition $\psi(t)<0$.

Remark 1. Theorem 1 contains, as a special case, the corresponding results of ([3]).

Remark 2. Other, in comparison with Theorem 1, conditions for the solvability of the Chaplygin problem for functional equations were considered by N. V. Azbelev and Z. B. Tsalyuk in the papers ([4]).

Consider the multipoint Vallée-Poussin problem ([5,6]) for the equation

$$x^n - p_n(t)x^{(n-1)} - \cdots p_1(t)x = 0, \tag{3}$$

$$x^{(l_i)}(a_i) = A_{il_i}, \quad i=1,2,\dots,m; \quad 2\le m\le n; \quad l_i = 0,1,\dots,r_i-1; \quad \sum_{i=1}^{m} r_i = n, \tag{4}$$

where $a\le a_1<\cdots<a_m$, $P=(p_1(t),\dots,p_n(t))$ are the coefficients of equation (3) continuous on the half-axis $[a,+\infty)$. The conditions for the solvability of the Chaplygin problem for problem (3), (4), as shown in ([7–9]), reduce to estimating the interval of nonoscillation of equation (3).

Suppose that the coefficients P satisfy the conditions

$$|p_i(t)|\le M_i, \quad i=1,2,\dots,n; \quad a\le t<+\infty, \tag{5}$$

$$(|P|\le M=(M_1,\dots,M_n)).$$

Let $\mu(P)$ be the exact upper bound of the right endpoints of all intervals of nonoscillation of equation (3) with common left endpoint at the point a, for fixed coefficients $P=(p_1(t),\dots,p_n(t))$ satisfying conditions (5). As is known ([8,10,11]), $[a,\mu(P))$ is the maximal interval of existence and uniqueness of the solution of problem (3), (4) for a fixed point a on the t-axis, fixed coefficients $P=(p_1(t),\dots,p_n(t))$ $(|P|\le M)$, and arbitrary numbers A_{ili}, m $(2\le m\le n)$, and arbitrary points a_i $(i=1,2,\dots,m;\ a\le a_1<\cdots<a_m<\mu(P))$. It is required to determine the quantity

$$\mu_0 = \inf_{|P|\le M}\mu(P).$$

Put $\alpha=(\delta_1,\dots,\delta_n)$, where δ_i $(i=1,2,\dots,n)$ is equal either to 0 or to 1; $\{\alpha_v\}$ $(v=1,2,3,\dots,2^n)$ is the set of all possible vectors $\alpha=(\delta_1,\dots,\delta_n)$; $\sigma=$

$(\alpha_1, \dots, \alpha_{f+1})$ is a collection of $f+1$ ($f = (n-1)n/2$) vectors α; $\{\sigma_u\}$ ($u = 1, 2, 3, \dots, 2^{n(f+1)}$) is the set of all possible collections σ, $\varphi_1^\alpha(t), \dots, \varphi_n^\alpha(t)$ ($\alpha = (\delta_1, \dots, \delta_n)$) is a fundamental system of solutions of the equation

$$x^n + (-1)^{\delta_n} M_{nx}^{(n-1)} + \dots + (-1)^{\delta_1} M_1 x = 0;$$

$$A_j^\alpha(t) = \|(\varphi_i^\alpha(t))^{(k)}\|, \qquad i = 1, 2, \dots, n; \quad k = 0, 1, \dots, j-1,$$

is a matrix having j ($j = 1, 2, \dots, n$) rows and n columns. Let $\Delta(t)$ be the determinant of the matrix

$$\left\| \begin{matrix} A_j^{\alpha_1}(a) & & & & 0 \\ A_n^{\alpha_1}(\tau_1) - A_n^{\alpha_2}(\tau_1) & & & & \\ & A_n^{\alpha_2}(\tau_2) - A_n^{\alpha_3}(\tau_2) & & & \\ & & \ddots & & \\ 0 & & & A_n^{\alpha_f}(\tau_f) - A_n^{\alpha_{f+1}}(\tau_f) & \\ & & & & A_{n-j}^{\alpha_{f+1}}(t) \end{matrix} \right\|, \qquad j = 1, 2, \dots, n-1,$$

where $\tau_1, \dots, \tau_f$ ($a \le \tau_1 \le \dots \le \tau_f < +\infty$) are f variable quantities; $t^* = t^*(j; \tau_1, \dots, \tau_f; \sigma_u)$ is the first root to the right of the point a of the equation $\Delta(t) = 0$, corresponding to some set of quantities $j; \tau_1, \dots, \tau_f; \sigma_u$ and satisfying the condition

$$t^*(j; \tau_1, \dots, \tau_f; \sigma_u) \ge \tau_f.$$

If the equation $\Delta(t) = 0$, for the given set of quantities $j; \tau_1, \dots, \tau_f; \sigma_u$, has no roots to the right of the point a satisfying the condition $t^*(j; \tau_1, \dots, \tau_f; \sigma_u) \ge \tau_f$, then we shall assume that

$$t^*(j; \tau_1, \dots, \tau_f; \sigma_u) = +\infty.$$

Theorem 2. Suppose that in problem (3), (4) conditions (5) are fulfilled (the conditions of the Vallée-Poussin theorem ([5,6])). Then, among all possible quantities

$$j = 1, 2, \dots, n-1; \quad \tau_1, \dots, \tau_f \ (a \le \tau_1 \le \dots \le \tau_f < +\infty); \quad u = 1, 2, 3, \dots, 2^{n(f+1)},$$

there will be found such $j_0; \tau_1^0, \dots, \tau_f^0; u_0$ that:

1)
$$t^*(j_0; \tau_1^0, \dots, \tau_f^0; \sigma_{u_0}) = \min_{j; \tau_1, \dots, \tau_f; u} t^*(j; \tau_1, \dots, \tau_f; \sigma_u);$$

2)
$$\mu_0 = \min_{j; \tau_1, \dots, \tau_f; u} t^*(j; \tau_1, \dots, \tau_f; \sigma_u) > a.$$

Remark 3. The quantity μ_0 cannot be improved under the conditions of Theorem 2 ([12,13]).

Remark 4. The assertion of Theorem 2 is preserved if, in conditions (5), the continuous coefficients $P = (p_1(t), \dots, p_n(t))$ of equation (3) are additionally assumed to be analytic on a finite interval (a, c), or are assumed measurable in (5).

Remark 5. The corresponding results of ([14–16]) are contained in Theorem 2 as a special case.

Consider the two-point Sturm-Liouville boundary-value problem

$$x'' - p_2(t)x' - p_1(t)x = 0; \tag{6}$$

$$\alpha_0 x(a) + \alpha_1 x'(a) = 0, \qquad \beta_0 x(b) + \beta_1 x'(b) = 0, \qquad \alpha_0 \neq 0,\ \beta_0 \neq 0, \tag{7}$$

where $P = (p_1(t), p_2(t))$ are coefficients of equation (6), continuous on the half-axis $[a, +\infty)$, satisfying the conditions

$$|p_i(t)| \leq M_i, \qquad i = 1, 2; \qquad a \leq t < +\infty$$

(abbreviated as $|P| \leq M = (M_1, M_2)$).

Assume that the condition of C. A. Pack is satisfied: $\Pi \neq 0$;

$$\Pi = \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}.$$

Denote by $\lambda(P)$ the exact upper bound of the set of all possible values of the quantity b $(b > a)$ satisfying the following condition: for fixed coefficients $P = (p_1(t), p_2(t))$, in the square $[a \leq t \leq b;\ a \leq s \leq b]$ the Green's function $G(t, s; P)$ of problem (6), (7) and its derivative $G'_t(t, s; P)$ retain their sign. It is required to determine the quantity

$$\lambda_0 = \inf_{|P| \leq M} \lambda(P).$$

Introduce the notation: $S_M = M_2^2 - 4M_1$;

$$t^0(\delta_2; \nu) = \min\big[t_1^0(\delta_2; \nu);\ t^+(0, \delta_2; \nu)\big];$$

$$t^-(\delta_2; \nu) = \min\big[t_1^-(\delta_2; \nu);\ t^+(0, \delta_2; \nu)\big];$$

$$t_0(S_M) = \min_{\nu} t(\nu; S_M);$$

$$t(\nu; S_M) = \begin{cases} \min\limits_{\delta_1, \delta_2} t^+(\delta_1, \delta_2; \nu), & \text{if } S_M > 0, \\ \min\limits_{\delta_2} t^0(\delta_2; \nu), & \text{if } S_M = 0, \\ \min\limits_{\delta_2} t^-(\delta_2; \nu), & \text{if } S_M < 0; \end{cases}$$

$$t^+(\delta_1, \delta_2; \nu) = \begin{cases} \dfrac{Q^+(\delta_1, \delta_2; \nu)}{k_1(\delta_1, \delta_2) - k_2(\delta_1, \delta_2)}, & \text{if } \operatorname{Re} Q^+(\delta_1, \delta_2; \nu) > 0 \text{ and} \\ & \operatorname{Im} Q^+(\delta_1, \delta_2; \nu) = 0; \\ +\infty, & \text{if at least } \operatorname{Re} Q^+(\delta_1, \delta_2; \nu) \leq 0 \text{ or} \\ & \operatorname{Im} Q^+(\delta_1, \delta_2; \nu) \neq 0; \end{cases}$$

$$t_1^0(\delta_2;\nu) = \begin{cases} (-1)^{\nu_0-1}Q^0(\delta_2;\nu), & \text{if } (-1)^{\nu_0-1}Q^0(\delta_2;\nu) > 0, \\ +\infty, & \text{if } (-1)^{\nu_0-1}Q^0(\delta_2;\nu) \le 0; \end{cases}$$

$$t_1^-(\delta_2;\nu) = \begin{cases} \dfrac{(-1)^{\nu_0-1}}{\eta}\operatorname{arctg} Q^-(\delta_2;\nu), & \text{if } (-1)^{\nu_0-1}Q^-(\delta_2;\nu) > 0, \\ \dfrac{(-1)^{\nu_0-1}}{\eta}\operatorname{arctg} Q^-(\delta_2;\nu) + \dfrac{\pi}{\eta}, & \text{if } (-1)^{\nu_0-1}Q^-(\delta_2;\nu) \le 0; \end{cases}$$

$$Q^+(\delta_1,\delta_2;\nu) = (-1)^{\nu_0}\ln\left\{\frac{\nu_1 + k_2(\delta_1,\delta_2)\nu_2}{\nu_1 + k_1(\delta_1,\delta_2)\nu_2}\left[\begin{matrix} 1 & \nu_3 \\ k_1(\delta_1,\delta_2) & \nu_4 \end{matrix}\Bigg/ \begin{matrix} 1 & \nu_3 \\ k_2(\delta_1,\delta_2) & \nu_4 \end{matrix}\right]\right\};$$

$$Q^0(\delta_2;\nu) = \frac{\nu_1\nu_3 + \nu_2\nu_4}{(\nu_4 - k(\delta_2)\nu_3)(\nu_1 + k(\delta_2)\nu_2)};$$

$$Q^-(\delta_2;\nu) = \frac{\nu_1\nu_3 + \nu_2\nu_4}{\nu_1(\nu_4 - \nu_3\xi(\delta_2)) + \nu_2[\xi(\delta_2)\nu_4 - \nu_3(\xi^2(\delta_2) + \eta^2)]};$$

$$\delta = (\delta_1,\delta_2); \quad \delta_i \ (i = 1,2) \text{ may take the values 0 or 1,}$$

and

$$\nu = (\nu_0,\nu_1,\nu_2,\nu_3,\nu_4)$$

takes the values

$$(0,1,0,-\alpha_1,\alpha_0), \quad (0,0,1,-\alpha_1,\alpha_0), \quad (1,1,0,-\beta_1,\beta_0),$$
$$(1,0,1,-\beta_1,\beta_0), \quad (0,\beta_0,\beta_1,-\alpha_1,\alpha_0);$$

$$k_{1,2}(\delta_1,\delta_2) = \frac{1}{2}\left[(-1)^{\delta_2}M_2 \pm \sqrt{M_2^2 + 4(-1)^{\delta_1}M_1}\right],$$

$$k(\delta_2) = \xi(\delta_2) = \frac{1}{2}(-1)^{\delta_2}M_2; \qquad \eta = \frac{1}{2}\sqrt{4M_1 - M_2^2}.$$

Theorem 3. Let, in problem (6), (7), the inequalities $|P| \le M$ and the conditions $\alpha_0 \ne 0$, $\beta_0 \ne 0$, $\Pi \ne 0$ be satisfied. Then there exist constant coefficients on the half-axis $[a,+\infty)$,

$$P^0 = ((-1)^{\delta_1^0}M_1,\ (-1)^{\delta_2^0}M_2)$$

of equation (6), such that

$$\lambda(P^0) = a + t_0(S_M) = \lambda_0 > a.$$

Let us now assume that in (7) $\gamma = -\alpha_1/\alpha_0 \ge 0$, $\alpha_0 \ne 0$;

$$\beta_0 = 0; \qquad \beta_1 \ne 0; \qquad \alpha_0 x(a) + \alpha_1 x'(a) = 0, \qquad x'(b) = 0. \tag{8}$$

From Theorem 3 there follows

Corollary 1. Let $\rho(P)$ be the exact upper bound of the right endpoints of all disconjugacy intervals ([17]) of problem (6), (8) with common left endpoint at the point a, corresponding to the given coefficients $P = (p_1(t), p_2(t))$ ($|P| \leq M$) of equation (6). Then there exist constant coefficients

$$\widetilde{P} = ((-1)^{\tilde{\delta}_1} M_1,\ (-1)^{\tilde{\delta}_2} M_2) \quad (\tilde{\delta}_1 = \tilde{\delta}_2),$$

such that

$$\rho(\widetilde{P}) = a + t(\tilde{\nu}; S_M) = \tilde{\rho} = \inf_{|P| \leq M} \rho(P) > a,$$

where

$$\tilde{\nu} = (0,\ 0,\ 1,\ -\alpha_1,\ \alpha_0).$$

Remark 6. Corollary 1 refines the corresponding results of S. A. Pak' s papers ([17a,b]), and for the quantities λ_0 and ρ remarks analogous to Remarks 3 and 4 are valid.

Consider the Sturm–Liouville problem for the nonlinear equation

$$x'' - f(t, x, x') = 0; \tag{9}$$

$$\alpha_0 x(a) + \alpha_1 x'(a) = d; \qquad \beta_0 x(b) + \beta_1 x'(b) = e. \tag{10}$$

We shall solve problem (9), (10) by Newton' s method ([1,18]):

$$x_{n+1}^{(i)}(t) = x_n^{(i)}(t) - \int_a^b G_{ti}^{(i)}(t, s; x_n)\,[x_n'' - f(s, x_n, x_n')]\,ds; \tag{11}$$

$$y_{n+1}^{(i)}(t) = y_n^{(i)}(t) - \int_a^b G_{ti}^{(i)}(t, s; x_n)\,[y_n'' - f(s, y_n, y_n')]\,ds \tag{12}$$

$$(i = 0, 1;\ n = 0, 1, ...),$$

where $G(t, s; x)$ is the Green' s function of the problem

$$g'' - \frac{\partial f(t, x, x')}{\partial x'}\, g' - \frac{\partial f(t, x, x')}{\partial x}\, g = h(t) \quad (h(t) \in C(a, b)),$$

$$\alpha_0 g(a) + \alpha_1 g'(a) = 0; \qquad \beta_0 g(b) + \beta_1 g'(b) = 0;$$

$x_0(t), y_0(t)$ are initial approximations to the solution of problem (9), (10), satisfying conditions (10). Put $c > a$; $\lambda = \min[t_0(s_m) t_0(S_M), c]$, $s_m = m_2^2 - 4m_1$; $S_M = M_2^2 - 4M_1$.

Theorem 4. Let in equation (9) the function $f(t, x, x')$ be twice continuously differentiable with respect to x and x' in the domain $a \leq t \leq c$; $-\infty < x < +\infty$; $-\infty < x' < +\infty$; let $x_0(t), y_0(t) \in C^2(a, b)$ satisfy (10) and the conditions:

1)
$$|\partial f(t,x_0,x_0')/\partial x_0| \le m_1; \quad |\partial f(t,x_0,x_0')/\partial x_0'| \le m_2;$$
$$|\partial f(t,y_0,y_0')/\partial y_0| \le M_1; \quad |\partial f(t,y_0,y_0')/\partial y_0'| \le M_2 \quad (a \le t \le c);$$

2) $a < b < \lambda$; $\Pi > 0$; $\alpha_0, \beta_0 < 0$; $\alpha_1, \beta_1 > 0$ $(\alpha_0, \beta_0 > 0;\ \alpha_1, \beta_1 < 0)$;

3) $x_0^{(i)}(t) \le y_0^{(i)}(t)$ $(i = 0, 1;\ a \le t \le b)$;

4)
$$x_0'' - f(t,x_0,x_0') \le 0 \le y_0'' - f(t,y_0,y_0') \quad (a \le t \le b);$$

5)
$$\partial^2 f(t,x,x')/\partial x^2 \ge 0; \quad \partial^2 f(t,x,x')/\partial x \partial x' \ge 0; \quad \partial^2 f(t,x,x')/\partial {x'}^2 > 0$$

for all t, x, x' from the domain $a \le t \le b$; $x_0(t) \le x \le y_0(t)$; $x_0'(t) \le x' \le y_0'(t)$.

Then there exists a unique solution $x^*(t) \in C^2(a,b)$ of problem (9), (10), satisfying the conditions

$$x_0^{(i)}(t) \le x^{*(i)}(t) \le y_0^{(i)}(t) \quad (i = 0, 1;\ a \le t \le b);$$

the sequences $\{x_n^{(i)}(t)\}$, $\{y_n^{(i)}(t)\}$ $(i = 0, 1;\ n = 0, 1, ...)$, defined by equalities (11), (12), converge uniformly on the interval $[a, b]$,

$$\lim_{n\to\infty} x_n^{(i)}(t) = \lim_{n\to\infty} y_n^{(i)}(t) = x^{*(i)}(t),$$

and the inequalities

$$x_n^{(i)}(t) \le x_{n+1}^{(i)}(t) \le x^{*(i)}(t) \le y_{n+1}^{(i)}(t) \le y_n^{(i)}(t);$$

$$x_n'' - f(t,x_n,x_n') \le 0 \le y_n'' - f(t,y_n,y_n')$$

hold $(i = 0, 1;\ n = 0, 1, \dots;\ a \le t \le b)$.

Remark 7. Theorem 4 refines the corresponding results ([18]).

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Note: Figure translations are in progress. See original paper for figures.