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BOUNDARY LAYER
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Abstract

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MATHEMATICS

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ASYMPTOTICS OF THE ERROR OF CUBATURE FORMULAS WITH A BOUNDARY LAYER REGULAR IN THE SENSE OF S. L. SOBOLEV

(Presented by Academician S. L. Sobolev on 28 III 1966)

The subject of the present note is the derivation of asymptotic estimates for the error of cubature formulas in the spaces $H_2^{(\mu)}(\Omega)$ (see ⁽⁶⁻⁸⁾). We have obtained such estimates under the assumption that the weight functions of these spaces belong to the class $B_n^{(m)}$, which we define as the collection of functions $\mu(\xi)$, defined in the whole space E^n , possessing there all derivatives up to order $2m$ inclusive and satisfying the conditions:

$$a^{-1}(1 + |\xi|^l)^{-1} \leq \mu(\xi) \leq a(1 + |\xi|^l); \quad (1)$$

$$\int_{E^n} \frac{1}{\mu^2(\xi)} d\xi < \infty; \quad (2)$$

$$\int_{E^n} \frac{1}{\mu^2(h^{-1}\xi)} d\xi \leq \frac{b^2}{\mu^2(h^{-1}e)} \int_{E^n} \frac{1}{\mu^2(\xi)} d\xi; \quad (3)$$

$$\int_{E^n} |\xi|^k \left| \Delta^m \left(\frac{1}{\mu^2(h^{-1}\xi)} \right) \right| d\xi \leq \frac{c_{k,m}^2}{\mu^2(h^{-1}e)}, \quad (4)$$

$$k = 0, 1, \dots, 2m,$$

where all constants and the unit vector e entering the inequalities (1)–(4) depend on the function $\mu(\xi)$ itself. In the case when $\mu(\xi)$ is specified at the points of some regular lattice, we shall assume that, instead of the integral inequalities, the corresponding inequalities for sums are fulfilled. Everywhere in what follows it is assumed that $2m > n$. The error functional of a cubature formula in

such spaces is linear and continuous; therefore the quality of the formula is conveniently characterized by the norm of this functional.

Computational practice shows that the nonuniformity of the distribution of nodes is often the principal source of error in approximate computation of integrals. Therefore, in what follows, we shall consider such cubature formulas whose error functionals satisfy certain regularity requirements.

Namely, let the error functional of a cubature formula be representable in the form

$$l(x) = \sum_{\gamma \in \Gamma} l_{h,\gamma}(x), \quad (5)$$

where all functionals $l_{h,\gamma}(x)$ satisfy the following conditions:

The nodes of all $l_{h,\gamma}(x)$ are located at the points of some regular lattice with principal period matrix hH

$$l_{h,\gamma}(x) = \mathcal{E}_{\Omega_{h,\gamma}}(x) - \sum_{|\gamma'| < L} C_{\gamma'}^{(\gamma)} \delta(x - hH\gamma') \quad (6)$$

($\Omega_{h,\gamma}$ is the domain obtained from the basic fundamental domain Ω_0 for the matrix H ($|H| = 1$) by means of translation and dilation);

$$\text{supp}(l_{h,\gamma}(x)) \subset \mathcal{E}(|x - hH\gamma| < Lh); \quad (7)$$

$$\|l_{h,\gamma}(x)\|_{C^*} \leq Ch^n; \quad (8)$$

$$(l_{h,\gamma}(x), x^\alpha) = 0, \quad \text{if } |\alpha| < m. \quad (9)$$

Cubature formulas whose error functionals satisfy all the requirements listed above we shall call **cubature formulas with a boundary layer regular in the sense of S. L. Sobolev**.

The boundary effect in such formulas is manifested in the fact that, as the nodes approach the boundary of the domain, a “smearing” of the corresponding coefficients occurs, whereas all coefficients corresponding to nodes lying sufficiently far from the boundary coincide with one another and are equal to h^n .

With a view toward obtaining error estimates, we shall consider first of all the periodic case, namely the case of the space $\widetilde{H}_2^{(\mu)}(\Omega)$. This is the space of periodic functions with principal period matrix hH , defined on a certain torus Ω , whose periods are multiples of the periods of the lattice $[hH]$. It is known ⁽³⁾ that

among all error functionals whose nodes are located at the points of the lattice $[hH]$, the optimal one is the functional of the form

$$l_{hH}^0(x) = 1 - \sum_{\gamma} h^n \delta(x - hH\gamma). \quad (10)$$

For it the following holds.

Theorem 1. *The norm of the optimal error functional of a cubature formula in the space $\widetilde{H}_2^{(\mu)}(\Omega)$ satisfies the inequality*

$$\|l_{hH}^0(x)\|_{\widetilde{H}_2^{(\mu)*}(\Omega)} \leq \frac{b}{\mu(h^{-1}e)} \sqrt{B_n^{(\mu)}(H)} \sqrt{|\Omega|}, \quad (11)$$

where

$$B_n^{(\mu)}(H) = \sum_{\gamma \neq 0} \frac{1}{\mu^2(\gamma H^{-1})}. \quad (12)$$

The proof is not difficult: just as in (8), one can show that

$$\|l_{hH}^0(x)\|_{\widetilde{H}_2^{(\mu)*}(\Omega)}^2 = \sum_{\gamma \neq 0} \frac{1}{\mu^2(\gamma h^{-1} H^{-1})} |\Omega|, \quad (13)$$

and our estimate follows immediately from the properties of the weight class.

In the general case of the space $H_2^{(\mu)}(\Omega)$, an asymptotically identical estimate holds. Namely, the following is true.

Theorem 2. *In the space $H_2^{(\mu)}(\Omega)$, the norm of the error functional of a cubature formula with a boundary layer regular in the sense of S. L. Sobolev satisfies the inequality*

$$\|l(x)\|_{H_2^{(\mu)*}(\Omega)} \leq \frac{b}{\mu(h^{-1}e)} \sqrt{B_n^{(\mu)}(H)} \sqrt{|\Omega|} + O\left(\frac{h}{\mu(h^{-1}e)}\right). \quad (14)$$

The proof is based on the following lemma:

Lemma. *Let the functionals $l_1(x)$ and $l_2(x)$ satisfy the following conditions:*

$$\text{supp}(l_1(x)) \subset \mathcal{E}(|x| < L_1), \quad \text{supp}(l_2(x)) \subset \mathcal{E}(|x| < L_2); \quad (15)$$

$$\|l_1(x)\|_{C^*} \leq A_1, \quad \|l_2(x)\|_{C^*} \leq A_2; \quad (16)$$

$$(l_1(x), x^\alpha) = 0, \quad \text{if } |\alpha| < k_1,$$

$$(l_2(x), x^\alpha) = 0, \quad \text{if } |\alpha| < k_2 \quad (k_1 + k_2 = 2m) \quad (17)$$

and let

$$v(x) = \int_{E^n} \frac{1}{\mu(\xi)} e^{-2\pi i \xi x} d\xi. \quad (18)$$

Then

$$\left| l_1\left(\frac{x}{h}\right) * v(x) * l_2\left(\frac{x}{h}\right) \right| \leq \begin{cases} B_1 \frac{h^n}{\mu^2(h^{-1}e)}, & \text{if } |x| \leq (L_1 + L_2)h, \\ B_2 \frac{h^{2m+n}}{\mu^2(h^{-1}e)|x|^{2m}}, & \text{if } |x| > (L_1 + L_2)h. \end{cases} \quad (19)$$

The first estimate is obtained directly. To obtain the second, one must represent $v(x)$ in the form

$$v(x) = \frac{(-1)^m}{|x|^{2m}} \int_{E^n} \Delta^m \left(\frac{1}{\mu^2(\xi)} \right) e^{-2\pi i \xi x} d\xi, \quad (20)$$

and then, taking into account that

$$\text{supp} \left(l_1\left(\frac{x}{h}\right) * l_2\left(\frac{x}{h}\right) \right) \subset \mathcal{E}(|x| \leq (L_1 + L_2)h),$$

expand $v(x - hy)$ in a Maclaurin series in the domain $|y| \leq (L_1 + L_2)h$:

$$v(x - hy) = \sum_{|\alpha| < 2m} \frac{D^\alpha v(x)}{\alpha!} (-h)^\alpha y^\alpha + R_{2m}(x, y), \quad (21)$$

where $R_{2m}(x, y)$, for $|x| > (L_1 + L_2)h$, satisfies the inequality

$$|R_{2m}(x, y)| \leq K \frac{h^{2m}}{h^n \mu^2(h^{-1}e) |x|^{2m}}. \quad (22)$$

The second estimate of the lemma follows from relations (17) and (22).

We now outline the proof of the main theorem. Represent the functional $l(x)$ in the form

$$l(x) = l_{hH}^0(x) - l'(x), \quad (23)$$

where $l'(x)$ is the error functional of a cubature formula with a regular boundary layer for the domain $\Omega' = E_n \setminus \Omega$. It is known ⁽⁸⁾ that the extremal function in the case of the space $H_2^{(\mu)}(\Omega)$ will be

$$u_m^0(x) = l(x) * v(x), \quad (24)$$

whence, by virtue of (23),

$$u_m^0(x) = u_{hH}^0(x) - (l'(x) * v(x)), \quad (25)$$

where $u_{hH}^0(x)$ is the extremal function of the periodic problem. Consequently,

$$\|l(x)\|_{H_2^{(\mu)*}(\Omega)}^2 = (l(x), u_{hH}^0(x)) - (l(x) * v(x) * l'(-x))|_{x=0}. \quad (26)$$

For the first term, by a direct calculation we obtain

$$(l(x), u_{hH}^0(x)) \leq \frac{b^2}{\mu^2(h^{-1}e)} B_n^{(\mu)}(H) |\Omega|. \quad (27)$$

For the second, using the lemma, we find the asymptotic behavior

$$(l(x) * v(x) * l'(-x))|_{x=0} = O\left(\frac{h}{\mu(h^{-1}e)}\right). \quad (28)$$

Theorem 2 follows directly from relations (26)–(28).

A direct consequence of Theorem 2 is

Theorem 3. *If there is a sequence of cubature formulas with a boundary layer regular in the sense of S. L. Sobolev, then the cubature process will converge for every function from the space $H_2^{(\mu)}(\Omega)$.*

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