

Investigation of computational algorithms for the approximate integration of differential equations using the finite difference method

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Abstract

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Preamble

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CRITIQUE AND BIBLIOGRAPHY**

P. S. Bondarenko. *Investigations of Computational Algorithms for the Approximate Integration of Differential Equations by the Finite Difference Method.* Kiev University Press, Kiev, 1962, 180 pp. (In Ukrainian).

The monograph by P. S. Bondarenko under review represents an interesting scientific investigation into a relatively understudied area of numerical methods in mathematics. The author introduces the concept of so-called “real” computational algorithms and provides a methodology for their construction and analysis. The concept of a real computational algorithm differs from the conventional notion (hereafter referred to as an “abstract” computational algorithm) in that every step of the real algorithm is assumed to be executed on a set of numbers with a fixed number of decimal places.

The monograph demonstrates that the theory of such real computational algorithms—which are the only ones encountered in actual computational practice—is characterized by a series of specific features that distinguish it significantly from the theory of abstract computational algorithms, which have been well-studied in the earlier works of other researchers. The book consists of a brief introduction and four chapters. In the first chapter, the author establishes relationships between differentiation operators and first-order difference operators, and also derives difference analogs of a priori estimates, such as the Poincaré and Friedrichs inequalities. These mathematical foundations serve as the basis

for his subsequent investigations into the convergence and stability of computational processes.

In the second chapter, the author introduces the concept of a real computational algorithm for the approximate integration of systems of ordinary differential equations and provides a definition of stability. He outlines a method for investigating the stability of real computational algorithms, which might appropriately be termed the “method of total majorants.” Consider a given difference scheme for integrating a differential equation consisting of a finite countable set of equations, where the solution of each is reduced to four arithmetic operations and logical operations. This process yields an “abstract” solution to the difference scheme—that is, a solution in which arithmetic operations are performed with exact precision. Alongside this, the author considers the “real” solution of the same difference scheme, where arithmetic operations are performed with rounding errors.

For the real solution of the difference scheme, the author constructs an auxiliary difference scheme whose abstract solution corresponds to the real solution of the original scheme. This real difference scheme incorporates not only the usual parameters (step length, approximation error estimates, etc.) but also an estimate of the computational error incurred at each step of the scheme. Using an explicit representation for the total deviation of the real solution from the corresponding abstract solution, the author proves a general criterion for the well-posedness of difference schemes for integrating ordinary differential equations and constructs majorants for the total deviation of both the abstract and real solutions from the exact solution of the original problem. As a corollary to these theorems, the author obtains the following result: for any well-posed difference scheme for integrating systems of ordinary differential equations, the total deviation of the real solution tends to zero as $\delta \rightarrow 0$ and does not grow faster than a finite power of h as $h \rightarrow 0$. This establishes the asymptotic convergence of the real approximate solution to the exact solution of the original problem. This result, obtained by P. S. Bondarenko as early as 1958 and published in 1959, was later independently arrived at by the Czech mathematician I. Babuška.

Thus, in the theory of stability for real computational algorithms, a theorem has been proven that is analogous to the well-known Lax-Richtmyer (or Filippov-Lax-Richtmyer) theorem, which states that the stability of a difference equation implies the convergence of its solution to the exact solution. The distinction here is that for a real approximate solution, stability guarantees asymptotic convergence. Among other results in the second chapter, one should note the algorithm for the numerical determination of the interval of existence for the solution to an initial value problem for systems of ordinary differential equations.

The author also establishes the unique solvability of the difference problem and the continuous dependence of its solution not only on small changes in boundary conditions and right-hand sides but also on small changes in the domain of the solution. With this result, Bondarenko identified the need to supplement the classical concept of well-posedness with the requirement of continuous

dependence on the domain. In the stability theorems for real computational algorithms, the property of coefficient stability of the equations is essentially utilized. From the proven theorems, the author specifically derives the following: for any well-posed difference scheme, the total deviation of its real solution from the abstract one tends to zero as $\delta \rightarrow 0$ and grows no faster than a finite power of h as $h \rightarrow 0$ (where δ represents the estimate of computational errors allowed in the real solution at the grid nodes). Thus, the logical convergence of the real approximate solution to the exact solution of the analytical task is established. After the author published this result in 1960, it was echoed by the American mathematicians B. Wendroff and G. Forsythe.

The nature of the convergence of the real solution for a well-posed difference scheme, as discovered by P. S. Bondarenko, allowed him to observe the unjustified restrictiveness of the stability condition in the sense of Lax-Richtmyer. This condition narrows the class of difference schemes actually suitable for the numerical integration of differential equations; the stability condition proposed by P. S. Bondarenko significantly expands this class. The same chapter presents multi-layer iterative methods for solving boundary value problems of elliptic type, which possess sufficiently good convergence, and proves a theorem on the asymptotic convergence of the real algorithm of successive approximations for solving systems of linear algebraic equations. These results are illustrated with thoroughly analyzed examples.

In Chapter IV, the general theory from Chapter III is applied to various problems in mathematical physics, the theory of thin plates, and the theory of elasticity, which were previously studied by S. G. Mikhailin using variational methods. P. S. Bondarenko's monograph—by its conception, its practical and theoretical importance, the novelty of the concepts introduced, and the rigorous analysis of convergence for both abstract and real computational algorithms—represents a significant contribution to science and serves as a valuable manual for computational mathematicians.

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Note: Figure translations are in progress. See original paper for figures.

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