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ON SOME PROPERTIES OF WEIGHT CLASSES

MATHEMATICS

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Abstract

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MATHEMATICS

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ON SOME PROPERTIES OF WEIGHT CLASSES

(Presented by Academician S. L. Sobolev, January 27, 1966)

1. Recently many authors have studied the properties and applications of the spaces of S. L. Sobolev with a weight. In the present note we touch upon two questions connected with the density of smooth functions in certain weighted spaces and with the characterization of spaces with zero traces by means of weighted integrals.

2. Denote by R_{n-1} the $(n-1)$-dimensional space of points $x' = (x_1, \dots, x_{n-1})$, and by R_n the space of points $x = (x', x_n)$. Let Q' be the unit cube in R_{n-1} with boundary $\partial Q'$. We shall consider functions $f(x)$ defined in the n-dimensional cube $Q = Q' \times (0,1)$ and equal to zero in a neighborhood of the set $\partial Q' \times (0,1)$ and of the hyperplane $x_n = 1$.

Let $\sigma = \sigma(t)$ be a nonnegative function defined on the interval $(0,1)$. Denote by $W^{(1)}_{p,\sigma}(Q)$, $1 \le p < \infty$, the set of functions $f(x)$ having finite norm

$$\|f\|_{W^{(1)}_{p,\sigma}(Q)} = \left\{ \int_Q \left[|f(x)\sigma(x_n)|^p + \sum_{i=1}^{n} \left| \frac{\partial f(x)}{\partial x_i} \sigma(x_n) \right|^p \right] dx \right\}^{1/p}. \tag{1}$$

Theorem 1. *Let $\sigma(t) \ge c > 0$ be a nondecreasing weight function for which*

$$\int_0^1 \sigma^p(t)\, dt < \infty, \tag{2}$$

$$\frac{1}{h\sigma^p(h)} \int_0^h \sigma^p(t)\, dt \le M \tag{3}$$

for sufficiently small $h > 0$. Then a function $f(x) \in W^{(1)}_{p,\sigma}(Q)$ can be approximated with arbitrary accuracy in the norm (1) by functions infinitely differentiable on $\overline{Q}$.

For the proof one constructs the function

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$$f_\varepsilon(x) = f_\varepsilon(x', x_n) = -\int_{x_n}^{1} \varphi_\varepsilon(x', t)\, dt,$$

where

$$\varphi_\varepsilon(x', t) = \begin{cases} 0, & \text{for } 0 < t < \varepsilon < 1, \\ \dfrac{\partial f}{\partial x_n}(x', t), & \text{for } t > \varepsilon. \end{cases}$$

The function $f_\varepsilon(x)$ belongs to the space $W^{(1)}_{p,\sigma}(Q)$ and approximates the function $f(x)$. Then the function $f_\varepsilon(x)$ is approximated by its averaging with an infinitely differentiable kernel of sufficiently small radius.

In the one-dimensional case $Q = (0, 1)$, condition (3) may be omitted.

Theorem 2. Let $\sigma(t) \geq c > 0$ be a nondecreasing weight function for which

$$\int_0^1 \sigma^p(t)\, dt = \infty. \tag{4}$$

Then the function $f \in W^{(1)}_{p,\sigma}(Q)$ can be approximated with arbitrary accuracy in the norm (1) by infinitely differentiable functions finite in Q.

From property (4) it follows that for almost all straight lines $x' = \text{const}$ there exists and is equal to zero

$$\lim_{x_n \to +0} f(x', x_n),$$

so that the function f can be extended by zero for $x_n < 0$. The function $f_\varepsilon(x', x_n) = f(x', x_n - \varepsilon)$ has compact support in Q and approximates the function $f(x', x_n)$ in the norm of the space $W^{(1)}_{p,\sigma}(Q)$. By means of averaging we obtain the desired infinitely differentiable finite function.

A similar assertion also holds for the spaces $W^{(k)}_{p,\sigma}(Q)$ when $k > 1$.

3. Let Ω be a bounded domain of n-dimensional space R_n with boundary $\partial\Omega$ locally satisfying the Lipschitz condition.

Let $\rho(x)$ be the distance from the point x to the boundary $\partial\Omega$, and let α be a real number, $1 \leq p < \infty$. Denote by $W^{(k)}_{p,\alpha}(\Omega)$ the space of functions with finite norm

$$\|f\|_{W^{(k)}_{p,\alpha}(\Omega)} = \left\{ \sum_{|i| \leq k} \int_\Omega |D^i f(x) \rho^\alpha(x)|^p\, dx \right\}^{1/p}. \tag{5}$$

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Let $C_0^\infty(\Omega)$ be the set of functions finite in Ω, and let $\overset{\circ}{W}{}^{(k)}_{p,\alpha}(Q)$ be its closure in the norm (5).

I. Nečas proved that for $\alpha \geq 0$ the set of functions infinitely differentiable on $\overline{\Omega}$ is dense in the space $W^{(k)}_{p,\alpha}(Q)$. From Theorem 1 it follows that a similar assertion holds for $k = 1$ also for

$$-1/p < \alpha < 0.$$

With the help of Theorem 2 one can show that for $\alpha \leq -1/p$

$$W^{(k)}_{p,\alpha}(\Omega) = \overset{\circ}{W}{}^{(k)}_{p,\alpha}(\Omega).$$

4. Let $\alpha = i - 1/p$ for $i = 1, \dots, k$. Denote by $V^{(k)}_{p,\alpha}(\Omega)$ the set of functions $f(x) \in W^{(k)}_{p,\alpha}(\Omega)$ for which

$$D^j f \in L_{p,\alpha-k+|j|}(\Omega), \qquad 0 \leq |j| \leq k.$$

If α coincides with one of the numbers $i - 1/p$, we define the space $V^{(k)}_{p,\alpha}(\Omega)$ somewhat differently. Let $\alpha = s - 1/p$, where s is an integer, $1 \leq s \leq k$. Further let $R > 0$ be sufficiently large. By $V^{(k)}_{p,\alpha}(\Omega)$ we shall understand the set of those functions $f \in W^{(k)}_{p,\alpha}(\Omega)$ for which

$$D^j f \in L_{p,\alpha-k+|j|}(\Omega) \qquad \text{for } k \geq |j| \geq k - s + 1,$$

$$\ln^{-1} \frac{R}{\rho} D^j f \in L_{p,\alpha-k+|j|}(\Omega) \qquad \text{for } k - s \geq |j| \geq 0,$$

where ρ is the distance to the boundary $\partial\Omega$.

Theorem 3. Let Ω be a domain with boundary locally satisfying the Lipschitz condition. Then for all real α the space

$$\overset{\circ}{W}{}^{(k)}_{p,\alpha}(\Omega) = V^{(k)}_{p,\alpha}(\Omega).$$

The embedding

$$\overset{\circ}{W}{}^{(k)}_{p,\alpha}(\Omega) \subset V^{(k)}_{p,\alpha}(\Omega)$$

is proved with the aid of the known Hardy inequality and some of its modifications. For the proof

of the inverse embedding one constructs a function $\varphi_\varepsilon(x) \in C_0^\infty(\Omega)$, $\varphi_\varepsilon(x) = 1$ for $\rho(x) \geq \varepsilon$, and such that

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$$|D^j \varphi_\varepsilon| \le \frac{c(j)}{\rho^{|j|}(x)} \frac{1}{\ln R/\rho}.$$

The function $f \in V_{p,\alpha}^{(k)}(\Omega)$ is approximated by means of $f\varphi_\varepsilon$ in the norm of the space $W_{p,\alpha}^{(k)}(\Omega)$. By averaging the function $f\varphi_\varepsilon$, we obtain a function from $C_0^\infty(\Omega)$.

Theorem 3 makes it possible to characterize functions with zero traces by means of spatial weighted integrals. Thus, for example, a function $f(x)$ is an element of the space $\mathring{W}_2^{(1)}(\Omega)$ if and only if $\partial f/\partial x_i \in L_2(\Omega)$ $(i = 1, 2, \dots, n)$ and $f/\rho \in L_2(\Omega)$.

With the aid of Theorem 3 one can prove that for $\alpha > k-1/p$ and for $\alpha \le -1/p$, $W_{p,\alpha}^{(k)}(\Omega) \equiv W_{p,\alpha}^{(k)}(\Omega)$. For $\alpha \le -1/p$ this assertion also follows from Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.

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