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THE STRUCTURE OF ELEMENTARY REPRESENTATIONS OF A SEMISIMPLE COMPLEX LIE GROUP

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I. M. Gel' fand and M. A. Naimark constructed ([1]), for each complex classical group G, a series of irreducible unitary representations, called by them the principal series, and showed that every function on the group G that is square-integrable with respect to Haar measure can be expanded in a Fourier integral with respect to the matrix elements of this series. These results were subsequently carried over by Harish-Chandra ([2]) to an arbitrary semisimple complex Lie group. In fact, the Gel' fand–Naimark formulas admit a certain analytic continuation with respect to the parameters, and as a result there arises a broader series of representations, which we shall agree to call **elementary**. It is known that not all elementary representations need be either irreducible or unitary. The author of the present note has studied in detail the elementary representations of the Lorentz group, and it was found ([3,4]) that among such representations there exist representations which do not have the property of complete reducibility; remarkable symmetry relations, holding for elementary representations, were also established. In the present note these results are generalized to an arbitrary semisimple complex connected group G.

Let us recall the description of elementary representations. Let D be a diagonal (Cartan) subgroup in the group G with canonical complex parameters $\lambda_1, \lambda_2, \dots, \lambda_r$ (the number r is called the rank of the group G). The formula

$$\alpha(\delta) = \exp\left(p_1\lambda_1 + \dots + p_r\lambda_r + q_1\overline{\lambda}_1 + \dots + q_r\overline{\lambda}_r\right)$$

defines a character of the group D if and only if all the numbers $p_i - q_i$ are integers.* For clarity we shall use the notation

$$\alpha(\delta) = \delta^p\overline{\delta}^q,$$

where p and q are pairs of vectors with coordinates $p_1, p_2, \dots, p_r$ and $q_1, q_2, \dots, q_r$, respectively. If one uses the Gauss decomposition

$$g = \zeta\delta z, \qquad \zeta \in Z_-, \quad \delta \in D, \quad z \in Z_+,$$

where Z_-, Z_+ are triangular subgroups in the group G, generated respectively by the negative and positive root vectors, then the function $\alpha(\delta)$ can, by the formula

$$\alpha(g) = \alpha(\delta)$$

be extended to the set Z_-DZ_+, which is everywhere dense in G. The question of continuation to the whole group G is decided as follows. If $p_i = 1$, and all the remaining coordinates p_j, q_j are zero, then the corresponding character (denote it by Δ_i) continues ([5]) to an analytic function $\Delta_i(g)$

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* The character $\alpha(\delta)$ is single-valued if G is simply connected (which may be assumed for convenience). In the general case $\alpha(\delta)$ is finitely valued.

on the group G. Accordingly,

$$\alpha(g) = \prod_{i=1}^{r} \Delta_i^{p_i - q_i}.$$

Every such function will be called an **elementary function** on the group G. If all the parameters are nonnegative integers, then the corresponding elementary function is connected in a definite way ([5,6]) with a finite-dimensional irreducible representation of the group G.

To each elementary function on the group G (and hence to each character $\alpha(\delta)$ of the group D) one can associate an infinite-dimensional representation of the group G in the following way. First of all, changing the numbering, we shall agree to write the elementary function in the form

$$\alpha(g) = \prod_{i=1}^{r} \Delta_i^{p_i - 1 - q_i - 1},$$

and at the same time to say that the function $\alpha(g)$ is determined by the signature

$$\alpha = (p, q).$$

Let $\mathfrak{D}_\alpha$ be the linear space of all infinitely differentiable functions on G satisfying the equation

$$\varphi(g_- g) = \alpha(g_-)\varphi(g),$$

where g_{-} ranges over the maximal solvable subgroup $G_{-} = ZD$. The representation of the group G, induced in $\mathfrak{D}_\alpha$ by the operators of right translation,

$$R_{g_0}\varphi(g) = \varphi(gg_0),$$

will be called the **elementary representation** of the group G with signature α.

If we note that $G = ZEu$, where E is the subgroup in G consisting of all positive definite diagonal matrices, and $\mathfrak{u}$ is the maximal compact subgroup, then it is clear that the functions in $\mathfrak{D}_\alpha$ are completely determined by their values on $\mathfrak{u}$; moreover, they satisfy the equation

$$\varphi(\gamma u) = \alpha(\gamma)\varphi(u),$$

where γ ranges over the maximal torus $\Gamma = D \cap \mathfrak{u}$, and the definition of the space $\mathfrak{D}_\alpha$ depends on the vector $m = q - p$, all of whose coordinates are integral (indeed, $\gamma^p\gamma^q = \gamma^{p-q} = \gamma^{-m}$). In this case the representation is given by the formula

$$T_g\varphi(u) = \varepsilon'^{\,\rho-2d}\varphi(u_g),$$

where ε', u_g are determined from the decomposition

$$ug = \zeta\varepsilon' u_g, \qquad \zeta \in Z_{-}, \quad \varepsilon' \in E, \quad u_g \in \mathfrak{u},$$

where the vector d has the form $(1, 1, \dots, 1)$, and $\rho = p + q$ is put.

We shall agree to call the vectors m and ρ, respectively, the **spectral base** and the **spectral indicator** of the signature α. We shall denote the representation obtained by $\varepsilon(\alpha)$. As is easy to verify, it is infinitely differentiable in the natural topology of $\mathfrak{D}_\alpha$ and can be extended to a continuous representation $\tilde{\varepsilon}(\alpha)$ in the Hilbert space H_α, obtained from $\mathfrak{D}_\alpha$ by completion in the metric

$$\|\varphi\|^2 = \int |\varphi(u)|^2\, du.$$

We now pass to the statement of the main results.

I. **Questions of irreducibility.** A representation T_g of the group G in a locally convex space E will be called **completely irreducible** if the weakly closed linear span of the operators T_g coincides

with the algebra $S(E)$ of all weakly continuous endomorphisms of the space E^*. The following assertions hold:

1. From complete irreducibility follows topological irreducibility.
2. If T_g is completely irreducible, then its contragredient $\widehat{T}_g = T_g'^{-1}$ is also completely irreducible.
3. Every weakly closed linear operator commuting with all operators of a completely irreducible representation is a multiple of the identity.

Let $P^+ = \{\omega_1, \omega_2, \dots, \omega_n\}$ be the system of all positive roots in the Lie algebra of the group G. We shall say that the signature α is positive (negative) with respect to ω_i if the numbers

$$p_i = 2(p, \omega_i)/(\omega_i, \omega_i), \qquad q_i = 2(q, \omega_i)/(\omega_i, \omega_i)$$

are positive (negative) integers. We shall say that α is neutral with respect to ω_i if either p_i, q_i are not integers, or $p_i q_i \leq 0$.

Theorem 1. The representation $e(\alpha)$, as well as $\tilde{e}(\alpha)$, is completely irreducible if and only if the signature α is neutral with respect to all positive roots.

Idea of the proof. Restricting first the representation $e(\alpha)$ to the subgroup U, we obtain the representation $e_0(\alpha)$, which is a discrete sum of finite-dimensional irreducible representations:

$$e_0(\alpha) = \sum_k n(k)\delta(k),$$

where $\delta(k)$ is the irreducible representation with highest weight k, and $n(k)$ is the corresponding multiplicity. It is easy to verify that $n(k)$ is equal to the multiplicity of the weight m in the representation $\delta(k)$, where m is the spectral base of the signature α. In particular, if k_0 is the minimal one among the highest weights, then $n(k_0) = 1$. The decisive step in our proof is the following

Fundamental lemma. Let e_j^k be a basis vector in $\mathfrak{D}_\alpha$ with highest weight k and number j, corresponding to the weight enumeration in $\delta(k)$ **. Put $f_0 = e_{j_0}^{k_0}$, where k_0 is the minimal one among the highest weights and j_0 is arbitrary. Then there exists such a polynomial $P_j^k(X)$ in the infinitesimal operators X of the group G that

$$P_j^k(X) f_0 = \left\{ \prod_{i=i_1}^{i_N} (\rho_i - a_i) \right\} e_j^k.$$

The polynomial P_j^k and the numbers $a_{i_1}, \dots, a_{i_N}$ $(i_1, \dots, i_N = 1, 2, \dots, n)$ depend only on the indices k and j. If $\rho_i = a_i$, then the signature α is positive with respect to the root ω_i.

Consequently, if α is negative or neutral with respect to all roots $\omega_i \in P^+$, then for each pair of indices k, j there is such a polynomial $F_j^k(X)$ that

$$F_j^k(X) f_0 = e_j^k.$$

Further, if α is positive or neutral with respect to all roots $\omega_i \in P^+$, then, passing to the conjugate representation, we find such a polynomial $\Phi_j^k(X)$ that

$$\Phi_j^k(X)e_j^k = f_0.$$

Let I_n be the finite-dimensional projector in $\mathfrak{D}_\alpha$ projecting onto all basis vectors e_j^k for which $k \leq n$. We see that, if α is neutral with respect to all roots $\omega_i \in P^+$, then the operators $I_{nP}(X)I_n$ fill the entire algebra of matrices $\nu(n)\times\nu(n)$, where $\nu(n) = \dim I_n$. But this is equivalent to the fact that $e(\alpha)$ $(\tilde{e}(\alpha))$ is completely irreducible. The theorem is proved.

II. **Double symmetry.** It is known about the representations of the "principal series" that they are equivalent under the replacement of $\alpha = (p, q)$ by $\alpha_s = (sp, sq)$, where

* This definition generalizes to locally convex spaces the well-known definition of R. Godement.

** In reality j is a pair of matrix indices with a definite choice of enumeration.

s is a transformation of the Weyl group. This can also be proved directly, by computing an operator W such that

$$We(a_s) = e(a)W.$$

The operator W turns out to be analytic in the parameters ρ and extends to all complex values of ρ, except for a certain family of exceptional points, where it can be regularized in a standard way. We call the operators W **operators of continuous symmetry**. All exceptional points of the operators W belong to the class of positive and negative ones. Further, it turns out that at these points there are still relations between elementary representations of the form

$$Se(a) = e(a')S,$$

where $\alpha' = (sp, \sigma q)$, $s \neq \sigma$, and S is a certain **operator of "discrete symmetry."** To each root $\omega_i \in P^+$ there corresponds a pair of such operators: $A_i = X_{+i}^{p_i}$, $B_i = X_{-i}^{q_i}$, where $X_{\pm i}$ are infinitesimal operators of the left shift on the group $\mathfrak{U}$. The operators A_i, B_i generate a system of operators S, and A_i, B_i correspond to the transition from $\alpha = (p, q)$ to $\alpha' = (sp, q)$, $\alpha'' = (p, sq)$, where s is the reflection from the Weyl group in the direction of the root ω_i.

III. **Structure of semisimple representations.** Analyzing the kernels and images of the symmetry operators, we obtain that the following is true.

Theorem 2. *Every elementary representation $e(\alpha)$ at an exceptional point α, positive or negative, has the form of a finite chain* $*$

$$e(\alpha) = \gamma_1 \to \gamma_2 \to \cdots \to \gamma_N,$$

where the representations γ_k are completely irreducible. Every invariant subspace in $e(\alpha)$ is an intersection of a finite number of kernels and images of operators of discrete symmetry.

Let us note that the number N depends on the "degree of degeneracy" of the signature α; in particular, if α is positive or negative only with respect to one root ω_i, and the numbers p_j, q_j for $j \neq i$ are not integers, then $N = 2$.

We shall now agree to regard the signatures α, α_s as equivalent. If the signature α is positive or neutral with respect to every root from P^+, then we denote by $\mu(\alpha)$ the restriction of $e(\alpha)$ to the cyclic envelope of the basis vectors with minimal highest weight k_0. ** Every representation $\mu(\alpha)$ will be called a **minimal representation of the group** G. From the proof of Theorem 1 it is not difficult to obtain that $\mu(\alpha)$ is completely irreducible.

Theorem 3. *To inequivalent signatures α there correspond inequivalent minimal representations $\mu(\alpha)$. Each of the representations $\gamma_1, \dots, \gamma_N$ under the conditions of Theorem 2 is equivalent to one of the minimal representations of the group G.*

Thus the representations $\mu(\alpha)$ are the true "elements" in the family $\{e(\alpha)\}$. In particular, if all the parameters p_i, q_i are nonnegative integers, the representation $\mu(\alpha)$ coincides with a finite-dimensional irreducible representation of the group G.

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* For the definition of chains, or "linkages," see (7).

** The space of the representation $\mu(\alpha)$ is singled out from $\mathfrak{D}_\alpha$ by a system of equations of the form

$$X_{+i}^{p_i} f = 0, \qquad X_{-i}^{q_i} f = 0, \qquad p_i > 0, \quad q_i > 0,$$

where p_i, q_i are all nonnegative integers from the signature α. If the signature α is neutral with respect to all roots, then $\mu(\alpha) = e(\alpha)$.

Note: Figure translations are in progress. See original paper for figures.