ON REMAINDER TERMS

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Abstract

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ON REMAINDER TERMS

IN MULTIDIMENSIONAL LIMIT THEOREMS

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At the present time several estimates are known for the rate of convergence of the probability function $P_n(A)$ of the normalized sum of independent identically distributed k-dimensional random vectors to the probability function $\Phi(A)$ of the k-dimensional normal law. When the set A is a k-dimensional interval, the best estimate is due to G. Bergström ([1]): if the summands have finite third moments, then

$$P_n(A) - \Phi(A) = O\left(\frac{1}{\sqrt{n}}\right).$$

When A belongs to the class of convex or Borel sets, it is known only that the remainder term has the form $O(n^{-1/2} \ln^{\alpha} n)$, $\alpha > 0$ (see ([2,3])). The natural question arises: under the same conditions, is it possible to get rid of the factor $\ln^{\alpha} n$? The first two theorems of the present paper are devoted to this question.

Consider a sequence of independent identically distributed k-dimensional random vectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \dots, \vec{\xi}^{(n)}$ of Euclidean space R_k. Suppose that $\vec{\xi}^{(1)}$ has finite moments of order s $(s > 0)$. Moreover, let $M\vec{\xi}^{(1)} = 0$.

We shall denote by $(\mathbf{x}, \mathbf{y})$ the scalar product of vectors $\mathbf{x}, \mathbf{y} \in R_k$; by $\mathfrak{A}$ the class of all Borel sets in R_k; $P_n(\mathbf{m}) = P\{\vec{\xi}^{(1)} + \dots + \vec{\xi}^{(n)} = \mathbf{m}\}$; $P_n(A)$ the probability function of the sum

$$S^{(n)} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \vec{\xi}^{(j)};$$

V the covariance matrix of the vector $\vec{\xi}^{(1)}$; $\lambda_1^{\nu_1} \lambda_2^{\nu_2} \dots \lambda_k^{\nu_k}$ the semi-invariants of order $(\nu_1, \nu_2, \dots, \nu_k)$ of the random vector $\vec{\xi}^{(1)}$. Following R. Rao ([2]), we define $\widetilde{P}_j(\vec{\omega})$ by means of the formal identity

$$\exp\left\{\sum_{j=3}^{\infty}\frac{(\vec{\lambda},\vec{\omega})^j}{j!}\left(\frac{1}{\sqrt{n}}\right)^{j-2}\right\}=\sum_{j=0}^{\infty}\left(\frac{1}{\sqrt{n}}\right)^{j}\widetilde{P}_j(\vec{\omega}),$$

and $P_j(-\varphi)(\mathbf{x})$ and $P_j(-\Phi)(\mathbf{x})$ by the equalities

$$P_j(-\varphi)(\mathbf{x})=\frac{1}{(2\pi)^k}\int_{\hat{R}_k}\widetilde{P}_j(i\mathbf{t})e^{-i(\mathbf{t},\mathbf{x})-\frac{1}{2}\mathbf{t}V\mathbf{t}'}\,d\mathbf{t},$$

$$P_j(-\Phi)(\mathbf{x})=\int_{\substack{y_1\le x_1\\ \dots\\ y_k\le x_k}}P_j(-\varphi)(\mathbf{y})\,d\mathbf{y}.$$

Theorem 1. *If the distribution function of the vector* $\vec{\xi}^{(1)}$ *has an absolutely continuous component, then*

$$\sup_{A\in\mathfrak{A}}\left|P_n(A)-\sum_{j=0}^{s-2}n^{-j/2}\int_A dP_j(-\Phi)\right|=o\left(n^{-(s-2)/2}\right).$$

Theorem 2. *If* $\vec{\xi}^{(1)}$ *has the identity covariance matrix and takes values in the lattice* $L=\{\mathbf{m}\}$, $\mathbf{m}=(m_1,m_2,\dots,m_k)$; $m_1,m_2,\dots,m_k=0,\pm1,\dots$, *with maximal span of the distribution equal to 1, then for all* k*, uniformly in* $A\in\mathfrak{A}$,

$$P_n(A)=\sum_{m/\sqrt{n}\in A}\sum_{j=0}^{s-2}n^{-(j+k)/2}P_j(-\varphi)\left(\frac{\mathbf{m}}{\sqrt{n}}\right)+o\left(n^{-(s-2)/2}\right). \qquad (1)$$

In the case where $\vec{\xi}^{(1)}$ has finite third moments, from (1), by means of the multidimensional Euler-Maclaurin summation formula, we obtain that for all k, uniformly in $A\in\mathfrak{A}$,

$$P_n(A)=\int_A dQ(\mathbf{x})+\frac{1}{\sqrt{n}}\int_A dP_1(-\Phi)+o\left(\frac{1}{\sqrt{n}}\right).$$

In particular, uniformly in $\mathbf{x}\in R_k$,

$$F_n(\mathbf{x})=\Phi(\mathbf{x})+\frac{1}{\sqrt{n}}P_1(-\Phi)(\mathbf{x})-\sum_{j=1}^{k}S(x_j\sqrt{n})\frac{\partial\Phi(\mathbf{x})}{\partial x_j}+o\left(\frac{1}{\sqrt{n}}\right).$$

Here $F_n(\mathbf{x})$ is the distribution function of the sum $S^{(n)}$; $\Phi(\mathbf{x})$ is the distribution function of the k-dimensional normal law with zero vector of mathematical expectations and identity covariance matrix,

$$Q(\mathbf{x}) = \prod_{j=1}^{k} \left[1 - \frac{1}{\sqrt{n}} S(x_j\sqrt{n}) \frac{\partial}{\partial x_j}\right] \Phi(\mathbf{x})$$

and $S(u) = [u] - u + \frac{1}{2}$, where $[u]$ is the integer part of u.

Theorems 1 and 2 for all $k < 2s$ are proved by the method of characteristic functions, and after this, with the help of the inequalities obtained, the proofs of the theorems for arbitrary k are completed by the method of truncation of random vectors.

Theorem 3. *Under the conditions of Theorem 2, for all* $k < 2s$,

$$\sum_{m \in L} \left| P_n(\mathbf{m}) - \sum_{j=0}^{s-2} n^{-(j+k)/2} P_j(-\varphi) \left(\frac{\mathbf{m}}{\sqrt{n}}\right) \right| = o\left(n^{-(s-2)/2}\right). \quad (2)$$

Let us note that relation (2) with a remainder term $O\left(n^{-(s-2)/2} \ln^{k/2} n\right)$ was obtained in ([2]).

Theorem 4. *Under the conditions of Theorem 2, for all* k,

$$\sum_{m \in L} \left|\frac{\mathbf{m}}{\sqrt{n}}\right|^{2\nu} \left| P_n(\mathbf{m}) - \sum_{j=0}^{s-2} n^{-(j+k)/2} P_j(-\varphi) \left(\frac{\mathbf{m}}{\sqrt{n}}\right) \right|^2 = o\left(n^{-s-k/2+2}\right)$$

for $\nu = 0, 1, \ldots, s$.

Theorems 3 and 4 are proved by the method of characteristic functions.

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Note: Figure translations are in progress. See original paper for figures.