ON BASES OF PRIMITIVE-RECURSIVELY CLOSED CLASSES OF FUNCTIONS

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Abstract

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ON BASES OF PRIMITIVE-RECURSIVELY CLOSED CLASSES OF FUNCTIONS

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Let K be an arbitrary primitive-recursively closed class of functions (([1])§4). We shall be interested in the question of in which classes K one can distinguish such a finite set of functions (a basis of K) that any function of the class K can be obtained by substitutions alone from the basic functions.

Let $K^{(1)}$ be the class of all one-place functions of the class K.

Lemma 1. *Every function of the class K can be obtained by substitutions alone from functions of the class $K^{(1)}$ and a certain two-place primitive-recursive function $\chi_0(x_1, x_2)$, constructed in* ([1]), *§ 4.*

Lemma 2. *If $K^{(1)}$ has a finite basis, then K also has a finite basis. Conversely, if K has a finite basis, then one can indicate such a finite set of functions from the class $K^{(1)}$ that any function in $K^{(1)}$ can be obtained from the functions of this set by means of a finite number of applications of the schemes*

$$\alpha(\beta(x)); \tag{1}$$

$$\chi_0(\alpha(x), \beta(x)). \tag{2}$$

Theorem 1. *The class K has a finite basis if and only if a function universal for the functions of $K^{(1)}$ belongs to K.*

Proof. Let the class K have a finite basis. Then let $\{f_1(x), \dots, f_n(x)\}$ be that set of functions from $K^{(1)}$ whose existence is asserted in Lemma 2.

Construct the function $\psi(y, x)$ in the following way:

$$\psi(y, x) =$$

$$= \begin{cases} f_1(x), & \text{if } y = 0, \\ f_2(x), & \text{if } y = 1, \\ \dots & \dots \\ f_n(x), & \text{if } y = n-1, \\ \psi(\exp_1(y); \psi(\exp_2(y); x)), & \text{if } y \geq n \text{ and } \exp_0(y) = 0, \\ \chi_0(\psi(\exp_1(y); x); \psi(\exp_2(y); x)), & \text{if } y \geq n \text{ and } \exp_0(y) > 0. \end{cases}$$

It is not difficult to show that $\psi(y, x)$ is a universal function for the class $K^{(1)}$. On the other hand, one can show that it is primitive-recursive relative to the functions $f_1(x), \dots, f_n(x)$. Consequently, $\psi(y, x) \in K$.

Conversely, let $\psi(y, x) \in K$, where $\psi(y, x)$ is a universal function for the functions of the class $K^{(1)}$. Then a basis for K will be the set of functions

$$\{0,\ x+1,\ \psi(y, x),\ \chi_0(x, y)\}.$$

Indeed, from the functions 0, $x+1$, $\psi(y, x)$, by substitutions one can construct all functions of the class $K^{(1)}$. The rest follows from Lemma 1.

Corollary 1. *The class of primitive-recursive functions $K_{p.r.}$ has no finite basis.*

Corollary 2. *The class of general recursive functions $K_{\text{o.r.}}$ has no finite basis.**

Corollary 3. *The class of partial recursive functions $K_{\text{p.r.}}$ has a finite basis.*

Corollary 4. *Let K be a primitive-recursively closed class containing only everywhere-defined functions. Then K has no finite basis.*

Suppose there is a finite set of functions

$$\{f_1, \dots, f_l\}. \tag{a}$$

For every natural number i we define, by induction on i, the notion of being a function of the i-th kind relative to (a). The class of functions of the i-th kind relative to (a) will be denoted by $R_i^{(a)}$.

Define $R_1^{(a)}$.

1. Every function from (a) belongs to $R_1^{(a)}$.
2. The result of substituting functions from $R_1^{(a)}$ into a function from $R_1^{(a)}$ is a function from $R_1^{(a)}$.

Suppose the class $R_i^{(a)}$ has been defined for some i. If i is even, then set:

1. Every function from $R_i^{(a)}$ is a function from $R_{i+1}^{(a)}$.

2. The result of substituting functions from $R_{i+1}^{(a)}$ into a function from $R_{i+1}^{(a)}$ is a function from $R_{i+1}^{(a)}$.

If i is odd, then set:

1. Every function from $R_i^{(a)}$ is a function from $R_{i+1}^{(a)}$.
2. Let $h(x)$ be a unary function from $R_i^{(a)}$; then the function $g(x)$, defined by the iteration scheme

$$g(0) = 0, \qquad g(x+1) = h(g(x)),$$

is a function from $R_{i+1}^{(a)}$.

Thus, the classes $R_i^{(a)}$ are defined for every i.

Remark 1. If one assumes that (a) contains the functions $x \dot{-} [\sqrt{x}]^2$, $x+1$, $x+y$, then every primitive recursive function is contained in all $R_i^{(a)}$, beginning with some ((3), §7).

Theorem 2. *For every finite set of primitive recursive functions (a) and for every natural number i, one can construct a primitive recursive function ψ_i such that $\psi_i \notin R_i^{(a)}$.*

Proof. It is known (3) that for every set (a) one can indicate such a set of primitive recursive functions

$$\{g_1(x), \dots, g_n(x), x+y\}, \tag{b}$$

which contains only one binary function $x+y$ and for which the inclusion holds for every i:

$$R_i^{(a)} \subseteq R_i^{(b)}.$$

In view of what has been said, it suffices to show that for each i there exists a primitive recursive function $\psi_i(y, x)$ possessing the following property: for every function $h(x_1, \dots, x_n) \in R_i^{(b)}$ one can indicate such a y_0 that

$$h(x_1, \dots, x_n) < \psi_i(y_0, \max(x_1, \dots, x_n)).$$

We construct $\psi_i(y, x)$ by induction on i. Put

$$\varphi(x) = \sum_{l=1}^{n} \sum_{j=0}^{x} g_l(j) + 2x + 2,$$

* Corollaries 1 and 2 for the classes $K^1_{\text{p.r.}}$ and $K^1_{\text{o.r.}}$ were obtained by another method in paper (2).

$$\psi_1(y,x) = \begin{cases} \psi_1(0,x) = \varphi(x), \\ \psi_1(y+1,x) = \psi_1(y, \psi_1(y,x)). \end{cases}$$

Suppose that $\psi_i(y,x)$ has already been defined for some i. Define $\psi_{i+1}(y,x)$. If i is even, put

$$\widetilde{\psi_i}(x) = \psi_i(x,x),$$

$$\psi^*_{i+1}(y,x) = \begin{cases} \psi^*_{i+1}(0,x) = \widetilde{\psi_i}^*(x), \\ \psi^*_{i+1}(y+1,x) = \psi^*_{i+1}(y, \psi^*_{i+1}(y,x)), \end{cases}$$

$$\psi_{i+1}(y,x) = \psi^*_{i+1}(y, \max(y,x)).$$

If i is odd, put

$$\psi_{i+1}(y,x) = \begin{cases} \psi_{i+1}(y,0) = \psi_i(y,0), \\ \psi_{i+1}(y,x+1) = \psi_i(y, \psi_{i+1}(y,x)). \end{cases}$$

It is not hard to show that the functions obtained are primitive-recursive and have the required properties.

Remark 2. Theorem 2 remains valid if, in defining the classes R'_i, iteration is replaced by the schema of primitive recursion.

Suppose there is a finite set (a) of general-recursive functions. Let

$$R^{(a)} = \bigcup_{i=1,2,\ldots} R^{(a)}_i.$$

Theorem 3. For every (a) one can indicate such a general-recursive function $\psi^{(a)}$ that

$$\psi^{(a)} \equiv R^{(a)}.$$

Theorem 4. If, in the construction of the classes $R^{(a)}_i$ (for even i), the operation of iteration is replaced by the general-recursive application of the operation μ ([1]), §7, then one can indicate such a finite set (a) of primitive-recursive functions that already $R^{(a)}_1$ coincides with the entire class of general-recursive functions.

The facts set forth above can be transferred to the theory of Turing machines. The class of all Turing machines, each of which computes some function, can be constructed by induction. As the initial machines we take machines computing the functions 0, $x+1$, and all argument-selection functions. We put, by definition, the number of places of each initial machine equal to the number of places of the function computed by the given machine. We now indicate operations which determine a new machine from machines already constructed.

Operation 1. Suppose machines $\mathfrak{M}_1^{(n)}, \dots, \mathfrak{M}_m^{(n)}, \mathfrak{M}_0^{(m)}$ have already been constructed (the upper index indicates the number of places). Construct the machine $\mathfrak{M}^{(n)}$ according to the schema (all notation is from ([4]), §68):

$$\mathfrak{M}^{(n)} = \mathfrak{K}_n \mathfrak{M}_1^{(n)} \mathfrak{I}_{n+1}^{n} \mathfrak{M}_2^{(n)} \cdots \mathfrak{I}_{n+1}^{n} \mathfrak{M}_m^{(n)} \mathfrak{I}_{(m-1)(n+1)+1} \mathfrak{I}_{(m-2)(n+1)+2} \cdots$$

$$\cdots \mathfrak{I}_{1\cdot(n+1)+(m-1)} \mathfrak{I}_{0\cdot(n+1)+m} \mathfrak{M}_0^{(m)} \mathfrak{L}\mathfrak{B}.$$

Operation 2. Suppose machines $\mathfrak{M}_1^{(n-1)}$, $\mathfrak{M}_2^{(n+1)}$ have already been constructed. Construct the machine $\mathfrak{M}^{(n)}$ according to the schema

$$\mathfrak{M}^{(n)} = \mathfrak{K}_n \mathfrak{M}_1^{(n-1)} \mathfrak{I}_{n+1} \mathfrak{C} \begin{cases} \mathfrak{I}_2 \mathfrak{L}\mathfrak{B}, \\ \mathfrak{C}\mathfrak{I}_3 \mathfrak{I}_{n+3}^{n-1} \mathfrak{M}_2^{(n+1)} \mathfrak{I}_{n+3} \mathfrak{T}\mathfrak{C} \begin{cases} \mathfrak{I}_2 \mathfrak{L}\mathfrak{B}, \\ \mathfrak{I}_{n+3}\mathfrak{A}. \end{cases} \end{cases}$$

Operation 3. Suppose the machine $\mathfrak{M}_1^{(n+1)}$ has been constructed. Construct the machine $\mathfrak{M}^{(n)}$ according to the scheme

$$\mathfrak{M}^{(n)} = \mathfrak{K}_n \mathfrak{C} \tilde{\mathfrak{S}}_{n+1}^{n} \dot{\mathfrak{M}}^{(n+1)} \mathfrak{C} \begin{cases} \tilde{\mathfrak{S}}_1 \mathfrak{L}\mathfrak{B}, \\ \tilde{\mathfrak{S}}_{n+2} \mathfrak{A} \tilde{\mathfrak{S}}_{n+2}^{n}. \end{cases}$$

Operations 1, 2, and 3 are, respectively, analogues of the operations of substitution, primitive recursion, and the μ-operation.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.