A general boundary value problem for elliptic systems of second order with constant coefficients. II

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Abstract

Full Text

Preamble

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A GENERAL BOUNDARY VALUE PROBLEM FOR SECOND-ORDER ELLIPTIC SYSTEMS WITH CONSTANT COEFFICIENTS

In this paper, we consider a general boundary value problem for a system of n second-order equations with constant coefficients of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0,$$

where A, B, C are constant square matrices of order n, and $u = (u_1, \dots, u_n)$ is the unknown vector.

The system is assumed to be elliptic in the sense of Petrovsky, meaning that the determinant of the characteristic matrix satisfies:

$$\det(A + B\lambda + C\lambda^2) \neq 0$$

for all real λ.

We investigate the problem in a simply connected domain D bounded by a smooth closed curve Γ. The boundary conditions are specified as:

$$\left(G\frac{\partial u}{\partial x} + H\frac{\partial u}{\partial y} + Qu\right)\Big|_{\Gamma} = f(s),$$

where G, H, Q are given constant matrices of size $n \times n$, and $f(s)$ is a given vector function of the arc length s on Γ.

The objective of this study is to establish the conditions for the Noethericity (Fredholm property) of this boundary value problem and to calculate its index. By employing the method of singular integral equations, we reduce the problem to a system of functional equations on the boundary.

1. Reduction to a Canonical Form

To analyze the system, we first utilize the algebraic properties of the characteristic polynomial. Since the system is elliptic, the roots of the equation $\det(A + B\lambda + C\lambda^2) = 0$ are all non-real. Let these roots be denoted by λ_k $(k = 1, \ldots, 2n)$. Without loss of generality, we assume that n roots

§ 4. ON THE NORMAL SOLVABILITY OF THE DIRICHLET PROBLEM

For system (1), reference [**?**] provides examples of elliptic systems for which the homogeneous Dirichlet problem in a disk possesses an infinite number of linearly independent solutions. This indicates that the Dirichlet problem for elliptic systems is, generally speaking, not Noetherian. However, the Dirichlet problem for system (1) may still be normally solvable, despite the fact that the homogeneous problem has infinitely many linearly independent solutions, as the definition of normal solvability does not require the number of linearly independent solutions to be finite. Consequently, the following questions naturally arise: 1. Is the Dirichlet problem for the elliptic system (1) always normally solvable? 2. Do there exist Dirichlet problems for elliptic systems that are normally solvable but not Noetherian? An answer to the second question is provided by the following example of a Dirichlet problem for an elliptic system (see [**?**]). Find a solution to the elliptic equation, regular in the unit disk $|z| < 1$:

$$u = u_1 + iu_2$$

satisfying the boundary condition

$$u|_{\Gamma} = h(x, y),$$

where $h(x, y)$ is a given complex-valued function continuous in the closed domain. It was proven in [**?**] that for this problem to be solvable, it is necessary and sufficient for the function $h(x, y)$ to satisfy a countable number of conditions of the form (81), where (D). It follows from Theorem 4 that this problem is normally solvable, even though the homogeneous problem has an infinite number of linearly independent solutions. Therefore, for the elliptic system (84), the Dirichlet problem is normally solvable but not Noetherian. The answer to the first question is negative. To demonstrate this, consider the following example: find a solution to the elliptic system, regular in the unit disk $|z| < 1$ (the first part of this work can be found in *Differential Equations*, No. 1, 1966):

belonging to the class and satisfying the boundary condition

and $M = 0$, (86)

where f is a given continuous function in $|z| < 1$, and a is a constant. Equation (85) represents an elliptic system of two differential equations with real coefficients. We shall prove that the homogeneous problem (85), (86) possesses no non-trivial solutions, and that the problem itself is not normally solvable. The general solution of the homogeneous equation (85) takes the form:

$$u = \phi(x + iy) + \psi(x + 2iy)$$

(87)

where $\phi(z)$ and $\psi(z)$ are analytic functions in the disk $|z| < 1$ and in the ellipse $x^2 + \frac{y^2}{4} < 1$, respectively. Let $u(x, y)$ be a solution to the homogeneous problem (85), (86). Then, according to (86) and (87), $u(x, y)$ can be represented in the form (87), where the functions ϕ and ψ satisfy the condition $\phi(x + iy) + \psi(x + 2iy) = 0$. Since $u \in C(D)$, it follows from (87) that $\phi(x+iy)$ and $\psi(x+2iy)$ also belong to the class $C(D)$. Let us denote $\psi_1(z) = \psi(z)$. Since $\psi(z)$ is analytic in the ellipse (88), the function $\psi_1(z)$ is analytic with respect to z in the annulus $\rho < |z| < 1$ and is continuous in the closed annulus $\rho \leq |z| \leq 1$. It is easily verified that

4 W H ' (- 7 T) '

According to (89), we obtain (2) = 0 when $|z| = 1$. It follows that $\phi(z)+\psi(z) = 0$ when $\frac{1}{L} < |z| < L$. Consequently, the equality (90) also holds for $\phi(z)$. By expanding the function $\phi(z)$ in a power series within the disk $|z| < 1$ and applying equality (90), we find that $\phi(z) = c$ (where c is a constant). From this and (91), we obtain $\psi_1(z) = -c$. Since $\psi_1(z) = \psi(\frac{1}{z})$, it follows that $\psi(x + iy) = -c$. Substituting the determined values of the functions ϕ and ψ into formula (87), we conclude that $u(x, y) = 0$ in the domain.

Remark 3. Similarly, it can be proven that the homogeneous problem (85), (86) possesses only the trivial solution even when this solution

satisfies the boundary condition (86) on an arbitrarily small part of the boundary, while on the remainder of the boundary, continuity of the desired solution is not even required. We shall now prove that problem (85), (86) is not normally solvable. According to Corollary 1, to prove this assertion, it is sufficient to show that for this problem, the function

3 ' - (z=x~iy) (92) 2(1 −ezf

has a solution if $0 < c < \frac{1}{3}$ and has no solution if $c = \frac{1}{3}$.

then it is easy to verify that the function $f(x) = \sqrt{x^2 + 3}$

$z = 2, \quad P = \sqrt{x^2 + 3},$

$= P$

a r g (: - M l 3i, p = a r g (^ - / - K ’ 3)) –< a , p < 2 * + –) 2 1 2 j

is a solution to the problem (85), (86) for any $h(x, y)$ of the form (92), where $z = x + iy$ and $\bar{z} = x - iy$. Let $c =$ .... We shall show that for $h(x, y)$ of the form (92), problem (85), (86) has no solution. Suppose, to the contrary, that problem (85), (86) for such an $h(x, y)$ has a solution $u_0(x, y)$. Then the function $v(x, y) = u(x, y) - u_0(x, y)$, given by formula (93), satisfies the homogeneous equation (85) in the domain $|z| < 1$ and the boundary condition (86) at all points of the boundary $|z| = 1$, except perhaps at the point $z = -1$. According to Remark 3, it follows that $u(x, y) = v(x, y) + u_0(x, y)$. However, since the function $u_0(x, y)$ is continuous at the point $z = -1$, while the function $u(x, y)$ has a singularity at this same point, equality (94) cannot hold. The resulting contradiction proves our assertion. From the examples provided, it is evident that the class of elliptic systems for which the Dirichlet problem is normally solvable is broader than the class of elliptic systems for which the problem is Noetherian, yet narrower than the entire class of elliptic systems. In [12], the following assertion was proved: if the characteristic equation (3) has an n-fold root λ, then for the Dirichlet problem for system (1) in a disk to be Noetherian, it is necessary and sufficient that system (1) be weakly coupled. This assertion remains valid regardless of the nature of the multiplicity of the roots in any finite domain with an analytical boundary and in the half-plane.

TOVMASYAN It should be noted that in a domain with an analytical boundary, the condition that the system be weakly coupled remains necessary for the Noetherian property of the Dirichlet problem even when the boundary data belong to the class of infinitely differentiable vector functions and the solution is sought in the class $C(\bar{D}) \cap C^2(D)$. A comparison of the results in the following section with the aforementioned assertion indicates that the class of elliptic systems for which the Dirichlet problem is normally solvable is much broader than the class of elliptic systems for which the Dirichlet problem is Noetherian. For the equation, the following theorem holds:

Theorem. For the Dirichlet problem of the system in a domain with an analytical boundary to be normally solvable, it is necessary and sufficient that the function $z(\zeta)$, which conformally maps the unit disk onto the domain, be rational.

This theorem implies that the normal solvability of the Dirichlet problem for strongly coupled systems depends very heavily on the configuration of the domain. Using the general solution of equation (84) (see [8]).

U = q(z)+~Z^(Z) + n) d g d 4 / :

If $\phi(z)$ and $\psi(z)$ are arbitrary analytic functions in the domain, the theorem can be proven quite simply. We omit the proof here.

The Dirichlet Problem for the System in the Case of a Half-Plane and a Disk

In this section, we present a simple method for solving the Dirichlet problem for the system in the case of a half-plane and a disk. For the disk, we consider the Dirichlet problem only for systems where the characteristic equation has an n-fold root.

The Dirichlet problem for the half-plane is considered in the following formulation:

Problem. Find a regular solution to the system in the half-plane $y > 0$ that belongs to the class C^0 in any closed semi-disk $|z| \leq R, y \geq 0$, and satisfies the following conditions:

u(x,0) = f(x), –oo < # < o o , • (96)

–represents the maximum order of the highest derivatives of the functions u_j appearing in the general solution (10). Regarding the vector-function $f(x)$ defined on the axis, it is assumed to belong to the class $C^{(l)}$ on any finite interval, and $g(x) = f(1/x)$ belongs to the class $C^{(l)}$ in the neighborhood of the point $x = 0$. In particular, finite functions satisfy these requirements. The following theorem holds:

Theorem. If the vectors a_r $(r = 1, \ldots, \nu)$ appearing in the general solution of the system are linearly independent, then the homogeneous problem $A(f = 0)$ has only the trivial solution, and the non-homogeneous problem is always solvable. If, however, these vectors are linearly dependent, then the homogeneous problem

Problem A has an infinite number of linearly independent solutions; for the solvability of the non-homogeneous problem A, it is necessary and sufficient that the vector-function $f(x)$ satisfies an infinite number of conditions $(f, \psi_k) = 0$ $(k = 1, 2, \ldots)$, where ψ_k is a set of linearly independent vectors. Here, c_{rj} $(r = 1, \ldots, \nu; j = 1, \ldots, n)$ are constant n-dimensional vectors that serve as linearly independent solutions to the algebraic system adjoint to the system $Lu = 0$.

Proof. By substituting the general solution (10) into the boundary condition (9b), we obtain

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w / r (^)) , = o - f W - c , (99)

$\Phi(z)$ is an analytical function in the half-plane Im $z > 0$. From (10), it follows directly that the expression

$\Phi_j^{(r-1)}(x+\lambda_j y)+\gamma_j^{(1)}\Phi_j^{(r-2)}(x+\lambda_j y)+\cdots+\gamma_j^{(r-1)}\Phi_j(x+\lambda_j y)y^{r-1}$

$(r=1,\dots,k_j;\ j=1,\dots,N_0)$

can be expressed linearly through the first derivatives of the desired solution. Consequently, $\Phi(z)$ is analytic in any semi-disk $\operatorname{Im} z>0$, and the following estimates hold: $|\Phi^{(r-1)}(z)|\le \text{const}\cdot\ln(2+|z|),\quad \operatorname{Im} z>0$ (100)

(r = 1, kj, / = 1, v 0).

From equations (99) and (100), we obtain:

$$\int_0^\infty \Phi(t,z)\phi(t)\,dt=f(z)-c \tag{101}$$

where c is a constant real vector. Let the vectors ξ_i $(i=1,\dots,n;\ j=1,\dots,\nu)$ be linearly independent. Then, by solving equation (101) with respect to $\phi_j(t)$ and substituting these solutions into (10), we obtain the desired solution to Problem L. To verify that this solution exhibits the required behavior at infinity, one must utilize the asymptotic estimate for large $|z|$:

$$|u(x,y)|\le\frac{C}{|z|^k}\quad(k=0,1,2,\dots,k_0). \tag{102}$$

The estimate (102) follows from the behavior of the given vector function $f(z)$ at infinity. From (10) and (101), the uniqueness of the solution to Problem L follows. Now, let us assume the vectors ξ_i $(i=1,\dots,n)$ are linearly independent. Let r denote the number of linearly independent vectors ξ_{ij} $(i=1,\dots,n; j=1,\dots,\nu)$. In this case, equation (101) possesses a solution if and only if $f(z)$ satisfies the conditions:

$$f(t)-f(\infty)=0\quad(k=1,\dots,n-r) \tag{103}$$

Condition (103) is equivalent to condition (97). When condition (97) is satisfied, the particular solution of system (101) takes the form:

$$\phi(t)=M^{-1}f(t) \tag{104}$$

where $\phi(t)=(\phi_1,\phi_2,\dots,\phi_\nu)^T$ and M is a well-defined $n\times\nu$ constant matrix. Substituting (104) into (10), we obtain a particular solution to Problem L. Consequently, condition (97) is both necessary and sufficient for the solvability of Problem L.

The solution to the homogeneous Problem L is given by formula (10), where $\Phi(z)$ is replaced by $Q_i(z)+\Phi(z)$; here, $a_1,\dots,a_{n-r}$ are n-dimensional constant vectors that serve as linearly independent solutions to the system (98). The functions $Q_1(z),\dots,Q_{n-r}(z)$ are arbitrary analytical functions in the upper half-plane $y>0$, chosen such that the solution $u(x,y)$ obtained via formula (10) maintains the required smoothness and behavior at infinity. In particular, $Q_i(z)$

can be taken in the form $Q_i(z) = (z - z_0)^{-k}$, where z_0 is an arbitrary fixed point in the lower half-plane and $k = 1, 2, \ldots$. Consequently, in this case, the homogeneous Problem L possesses an infinite number of linearly independent solutions.

Remark:

3. Theorem 7 remains valid (only slightly changing)

The condition of the form (92) remains valid even if the half-plane $y > 0$ is replaced by any other half-plane in the (x, y) plane. Let us now consider the Dirichlet problem for system (1) in the disk $|z| < 1$ under the following formulation: find a regular solution to the equation in the disk D belonging to the class $C^1(\bar{D})$ and satisfying the boundary condition:

$$u|_{|z|=1} = f(z) \quad (105)$$

where $f(z)$ is a given vector function from the class C^α. We shall assume that λ is an n-fold root of the characteristic equation (3). Then the general solution (10) in the disk can be rewritten in the form:

$$u(x, y) = \operatorname{Re}\left(\sum_{k=1}^{n} \delta_k \phi_k(z) + \sum_{k=2}^{n} \delta_k \sum_{j=1}^{k-1} \frac{(z - \bar{z})^j}{j!} \phi_k^{(j)}(z)\right) \quad (106)$$

where $\phi_k(z)$ are arbitrary analytic functions in the disk $|z| < 1$.

Substituting the general solution (106) into the boundary condition (105), we obtain:

$$\operatorname{Re}\left(\sum_{j=1}^{n} \delta_j \psi_j(z)\right)_{|z|=1} = f(z) \quad (107)$$

where

$$\psi_j(z) = \phi_j(z) + \sum_{k=j+1}^{n} \frac{(z - \bar{z})^{k-j}}{(k-j)!} \phi_k^{(k-j)}(z) \quad (108)$$

$(j = 1, 2, \ldots, n)$.

It is clear that $\psi_j(z)$ are analytic functions that may have poles at the point $z = 0$ of order no higher than $n - 1$. First, we find the solutions $\psi_1, \ldots, \psi_n$ of the boundary value problem (107) that are analytic in the disk $|z| < 1$ and have poles at $z = 0$ of order at most $n - 1$. From (107), we have:

$$\sum_{j=1}^{n} \delta_j \psi_j(z) = \Phi(z) + \sum_{j=1}^{n-1} \frac{c_j}{z^j} \quad (109)$$

where c_j $(j = 1, \ldots, n-1)$ are constant vectors, and the purely imaginary vector c_n is defined accordingly. Let the number of linearly independent vectors δ_j be

r. Then (109) has a solution in the considered class if and only if the following conditions are satisfied:

$$\operatorname{Re}\int_{|z|=1}(f,\chi_k)ds=0 \quad (110)$$

where $k=1,\dots,n-r$, and χ_k are linearly independent solutions of the linear homogeneous algebraic system adjoint to the system:

$$\sum_{j=1}^{n}\delta_j x_j=0 \quad (111)$$

When these conditions are met, the solution to system (109) is given by the formula:

$$\psi(z)=\mathbf{A}\Phi(z)+\sum a_j\omega_j+\sum_{k=1}^{n-1}\frac{g_k}{z^k} \quad (112)$$

where $\mathbf{A}$ is a well-defined constant n-dimensional matrix constructed using the vectors $\delta_1,\dots,\delta_n$; ω_j are linearly independent solutions of the algebraic system (111); and g_k are arbitrary vectors. The condition can be written as $j=1,\dots,p$ (113).

TOVMASYAN $(j=1,\dots,p)$ (114).

$(f_k=1,n-r_0,j=1,n-1)$.

The analytic functions $Q_i(z)$ appearing in (112) can be represented as $(j=1,\dots,p)$ (116), where $h=-k+1$ are arbitrary analytic functions in the domain $|z|<R$. Let the vector-function $\psi(z)$ and the vectors g_i satisfy conditions (113)-(115). Then, by substituting the value of ψ from (112) into (108) and solving (108) with respect to $(\phi_1(z),\dots,\phi_n(z))$, we find that this solution is regular at zero if and only if a finite number of conditions of the following form are satisfied:

$$\sum a_jM_j+\int(g_i,\Theta_i)dz=0, \quad (117)$$

where a_j are constant numbers, g_i are constant vectors, and Θ_i are vector-functions. These quantities are fully determined. It is clear that part of the conditions (114), (115), and (117) can be satisfied using the vectors g_i and the numbers a_j; the remaining part of the conditions can be replaced by conditions of the form:

$$(f,\chi_k)=d_k \quad (k=1,2,\dots,N). \quad (118)$$

Condition (110) together with (118) is necessary and sufficient for the solvability of the Dirichlet problem considered here in the disk. If conditions (110) and (118) are fulfilled, the solution to the problem is given by formula (106), where $\Phi(z)=(\phi_1(z),\dots,\phi_n(z))$ is an arbitrary solution of system (107); $\psi(z)$ is determined by (112); Q is given by (116); g_i are arbitrary vectors (where g is a real vector); and a_j are arbitrary numbers satisfying conditions (114), (115), and (117).

We note that the Dirichlet problem for weakly coupled systems in a disk was solved by a slightly different, yet similarly simple method in work [?]. Using the results obtained, it is easy to prove that the Dirichlet problem for the inhomogeneous system (1) with homogeneous boundary data in the case of a half-plane or a disk is always normally solvable. The problems considered in this section are related to the works of A. Dzuraev [?] and Ya. B. Lopatinskii [?].

[?] A. Dzuraev, Doklady AN Tadzhikskoi SSR, vol. VII, no. 4, 1964. [?] Ya. B. Lopatinskii, Ukr. matem. zhurnal, no. 2, 1959.## § 5 Conclusion

The case where $n = 2$ and the value represents a double root of the characteristic equation finds significant application in the plane theory of elasticity.

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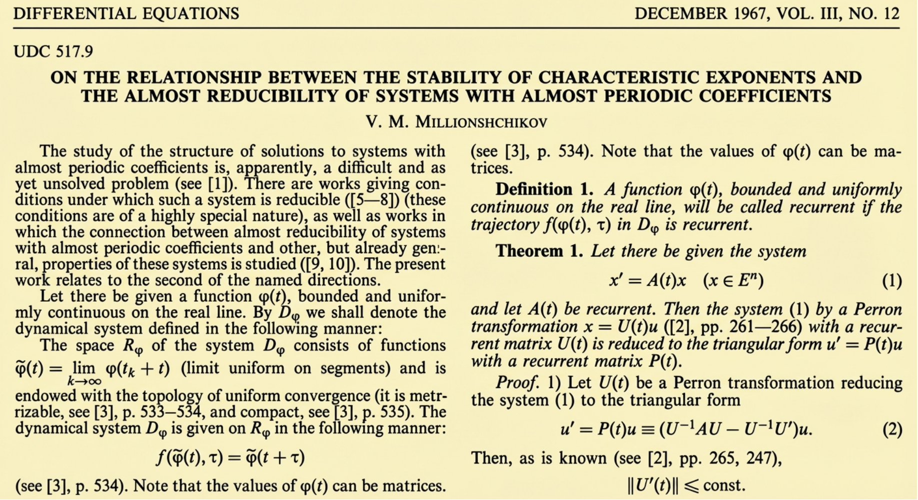
Figures



DIFFERENTIAL EQUATIONS DECEMBER 1967, VOL. III, NO. 12

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ON THE RELATIONSHIP BETWEEN THE STABILITY OF CHARACTERISTIC EXPONENTS AND THE ALMOST REDUCIBILITY OF SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

V. M. Millionshchikov

The study of the structure of solutions to systems with almost periodic coefficients is, apparently, a difficult and as yet unsolved problem (see [1]). There are works giving conditions under which such a system is reducible ([5—8]) (these conditions are of a highly special nature), as well as works in which the connection between almost reducibility of systems with almost periodic coefficients and other, but already general, properties of these systems is studied ([9, 10]). The present work relates to the second of the named directions.

Let there be given a function $\varphi(t)$, bounded and uniformly continuous on the real line. By D_φ we shall denote the dynamical system defined in the following manner:

The space R_φ of the system D_φ consists of functions $\tilde{\varphi}(t) = \lim\limits_{k\to\infty} \varphi(t_k + t)$ (limit uniform on segments) and is endowed with the topology of uniform convergence (it is metrizable, see [3], p. 533–534, and compact, see [3], p. 535). The dynamical system D_φ is given on R_φ in the following manner:

$$f(\tilde{\varphi}(t), \tau) = \tilde{\varphi}(t + \tau)$$

(see [3], p. 534). Note that the values of $\varphi(t)$ can be matrices.

(see [3], p. 534). Note that the values of $\varphi(t)$ can be matrices.

Definition 1. *A function $\varphi(t)$, bounded and uniformly continuous on the real line, will be called recurrent if the trajectory $f(\varphi(t), \tau)$ in D_φ is recurrent.*

Theorem 1. *Let there be given the system*

$$x' = A(t)x \quad (x \in E^n) \tag{1}$$

and let $A(t)$ be recurrent. Then the system (1) *by a Perron transformation $x = U(t)u$* ([2], pp. 261—266) *with a recurrent matrix $U(t)$ is reduced to the triangular form $u' = P(t)u$ with a recurrent matrix $P(t)$.*

Proof. 1) Let $U(t)$ be a Perron transformation reducing the system (1) to the triangular form

$$u' = P(t)u \equiv (U^{-1}AU - U^{-1}U')u. \tag{2}$$

Then, as is known (see [2], pp. 265, 247),

$$\|U'(t)\| \leqslant \text{const}.$$

Figure 1: Figure 1

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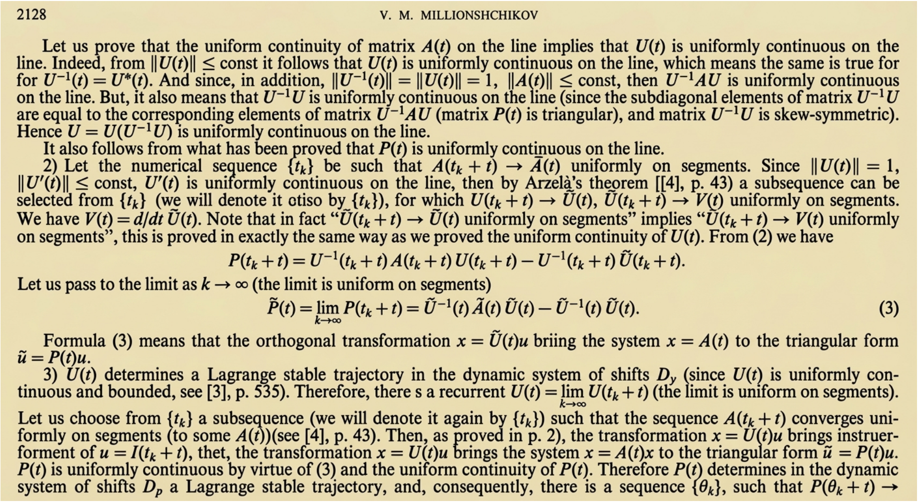

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Let us prove that the uniform continuity of matrix $A(t)$ on the line implies that $U(t)$ is uniformly continuous on the line. Indeed, from $\|U(t)\| \leq \text{const}$ it follows that $U(t)$ is uniformly continuous on the line, which means the same is true for for $U^{-1}(t) = U^{*}(t)$. And since, in addition, $\|U^{-1}(t)\| = \|U(t)\| = 1$, $\|A(t)\| \leq \text{const}$, then $U^{-1}AU$ is uniformly continuous on the line. But, it also means that $U^{-1}U$ is uniformly continuous on the line (since the subdiagonal elements of matrix $U^{-1}U$ are equal to the corresponding elements of matrix $U^{-1}AU$ (matrix $P(t)$ is triangular), and matrix $U^{-1}U$ is skew-symmetric). Hence $U = U(U^{-1}U)$ is uniformly continuous on the line.

It also follows from what has been proved that $P(t)$ is uniformly continuous on the line.

2) Let the numerical sequence $\{t_k\}$ be such that $A(t_k + t) \to \bar{A}(t)$ uniformly on segments. Since $\|U(t)\| = 1$, $\|U'(t)\| \leq \text{const}$, $U'(t)$ is uniformly continuous on the line, then by Arzelà's theorem [[4], p. 43) a subsequence can be selected from $\{t_k\}$ (we will denote it otiso by $\{t_k\}$), for which $U(t_k + t) \to \tilde{U}(t)$, $\tilde{U}(t_k + t) \to V(t)$ uniformly on segments. We have $V(t) = d/dt\, \tilde{U}(t)$. Note that in fact "$\tilde{U}(t_k + t) \to \tilde{U}(t)$ uniformly on segments" implies "$\tilde{U}(t_k + t) \to V(t)$ uniformly on segments", this is proved in exactly the same way as we proved the uniform continuity of $U(t)$. From (2) we have

$$P(t_k + t) = U^{-1}(t_k + t)\, A(t_k + t)\, U(t_k + t) - U^{-1}(t_k + t)\, \tilde{U}(t_k + t).$$

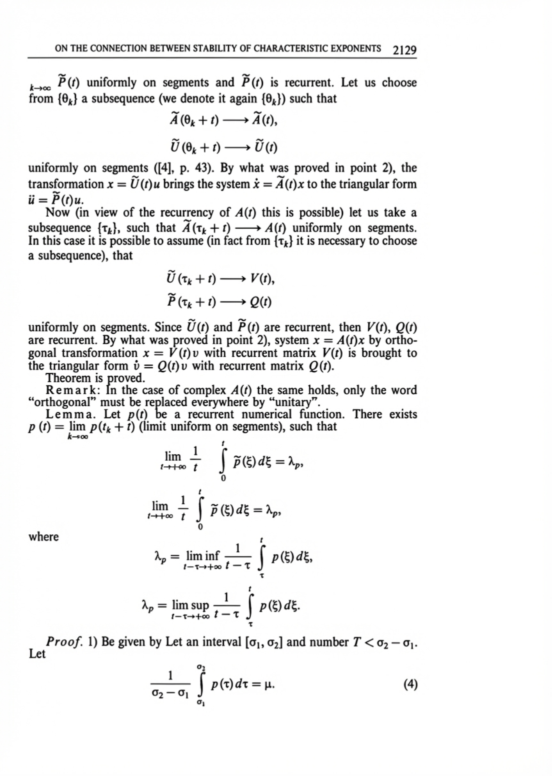
Let us pass to the limit as $k \to \infty$ (the limit is uniform on segments)

$$\tilde{P}(t) = \lim_{k\to\infty} P(t_k + t) = \tilde{U}^{-1}(t)\, \tilde{A}(t)\, \tilde{U}(t) - \tilde{U}^{-1}(t)\, \tilde{U}(t). \tag{3}$$

Formula (3) means that the orthogonal transformation $x = \tilde{U}(t)u$ briing the system $x = A(t)$ to the triangular form $\tilde{u} = P(t)u$.

3) $U(t)$ determines a Lagrange stable trajectory in the dynamic system of shifts D_y (since $U(t)$ is uniformly continuous and bounded, see [3], p. 535). Therefore, there s a recurrent $U(t) = \lim_{k\to\infty} U(t_k + t)$ (the limit is uniform on segments). Let us choose from $\{t_k\}$ a subsequence (we will denote it again by $\{t_k\}$) such that the sequence $A(t_k + t)$ converges uniformly on segments (to some $A(t)$)(see [4], p. 43). Then, as proved in p. 2), the transformation $x = U(t)u$ brings instruerforment of $u = I(t_k + t)$, thet, the transformation $x = U(t)u$ brings the system $x = A(t)x$ to the triangular form $\tilde{u} = P(t)u$. $P(t)$ is uniformly continuous by virtue of (3) and the uniform continuity of $P(t)$. Therefore $P(t)$ determines in the dynamic system of shifts D_p a Lagrange stable trajectory, and, consequently, there is a sequence $\{\theta_k\}$, such that $P(\theta_k + t) \to$

Figure 2: Figure 2



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$_{k\to\infty}$ $\tilde{P}(t)$ uniformly on segments and $\tilde{P}(t)$ is recurrent. Let us choose from $\{\theta_k\}$ a subsequence (we denote it again $\{\theta_k\}$) such that

$$\tilde{A}(\theta_k + t) \longrightarrow \tilde{A}(t),$$

$$\tilde{U}(\theta_k + t) \longrightarrow \tilde{U}(t)$$

uniformly on segments ([4], p. 43). By what was proved in point 2), the transformation $x = \tilde{U}(t)u$ brings the system $\dot{x} = \tilde{A}(t)x$ to the triangular form $\ddot{u} = \tilde{P}(t)u$.

Now (in view of the recurrency of $A(t)$ this is possible) let us take a subsequence $\{\tau_k\}$, such that $\tilde{A}(\tau_k + t) \longrightarrow A(t)$ uniformly on segments. In this case it is possible to assume (in fact from $\{\tau_k\}$ it is necessary to choose a subsequence), that

$$\tilde{U}(\tau_k + t) \longrightarrow V(t),$$

$$\tilde{P}(\tau_k + t) \longrightarrow Q(t)$$

uniformly on segments. Since $\tilde{U}(t)$ and $\tilde{P}(t)$ are recurrent, then $V(t)$, $Q(t)$ are recurrent. By what was proved in point 2), system $x = A(t)x$ by orthogonal transformation $x = V(t)\upsilon$ with recurrent matrix $V(t)$ is brought to the triangular form $\dot{\upsilon} = Q(t)\upsilon$ with recurrent matrix $Q(t)$.

Theorem is proved.

Remark: In the case of complex $A(t)$ the same holds, only the word "orthogonal" must be replaced everywhere by "unitary".

Lemma. Let $p(t)$ be a recurrent numerical function. There exists $p(t) = \lim_{k\to\infty} p(t_k + t)$ (limit uniform on segments), such that

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \tilde{p}(\xi)\, d\xi = \lambda_p,$$

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \tilde{p}(\xi)\, d\xi = \lambda_p,$$

where

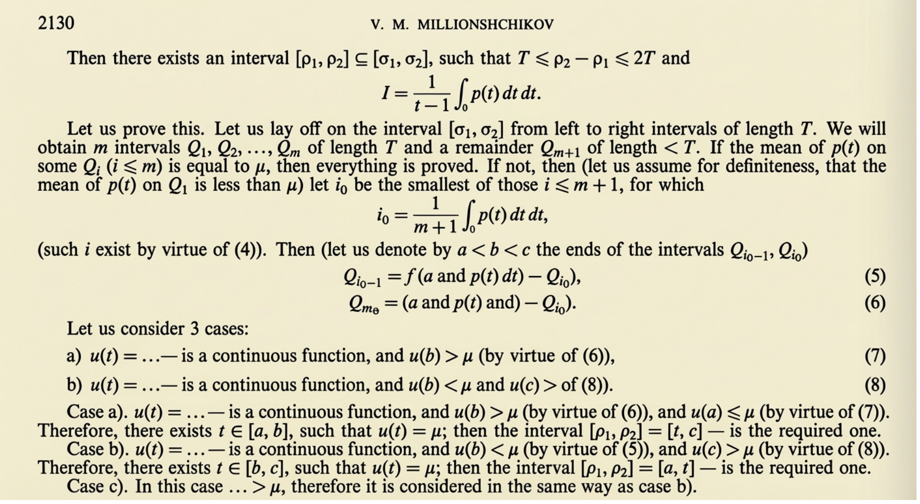
$$\lambda_p = \liminf_{t-\tau\to+\infty} \frac{1}{t-\tau} \int_\tau^t p(\xi)\, d\xi,$$

$$\lambda_p = \limsup_{t-\tau\to+\infty} \frac{1}{t-\tau} \int_\tau^t p(\xi)\, d\xi.$$

Proof. 1) Be given by Let an interval $[\sigma_1, \sigma_2]$ and number $T < \sigma_2 - \sigma_1$. Let

$$\frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} p(\tau)\, d\tau = \mu. \tag{4}$$

Figure 3: Figure 3



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Then there exists an interval $[\rho_1, \rho_2] \subseteq [\sigma_1, \sigma_2]$, such that $T \leqslant \rho_2 - \rho_1 \leqslant 2T$ and

$$I = \frac{1}{t-1} \int_0 p(t)\, dt\, dt.$$

Let us prove this. Let us lay off on the interval $[\sigma_1, \sigma_2]$ from left to right intervals of length T. We will obtain m intervals $Q_1, Q_2, \ldots, Q_m$ of length T and a remainder Q_{m+1} of length $< T$. If the mean of $p(t)$ on some Q_i $(i \leqslant m)$ is equal to μ, then everything is proved. If not, then (let us assume for definiteness, that the mean of $p(t)$ on Q_1 is less than μ) let i_0 be the smallest of those $i \leqslant m+1$, for which

$$i_0 = \frac{1}{m+1} \int_0 p(t)\, dt\, dt,$$

(such i exist by virtue of (4)). Then (let us denote by $a < b < c$ the ends of the intervals Q_{i_0-1}, Q_{i_0})

$$Q_{i_0-1} = f(a \text{ and } p(t)\, dt) - Q_{i_0}), \tag{5}$$

$$Q_{m_\Theta} = (a \text{ and } p(t) \text{ and}) - Q_{i_0}). \tag{6}$$

Let us consider 3 cases:

a) $u(t) = \ldots$ — is a continuous function, and $u(b) > \mu$ (by virtue of (6)), (7)

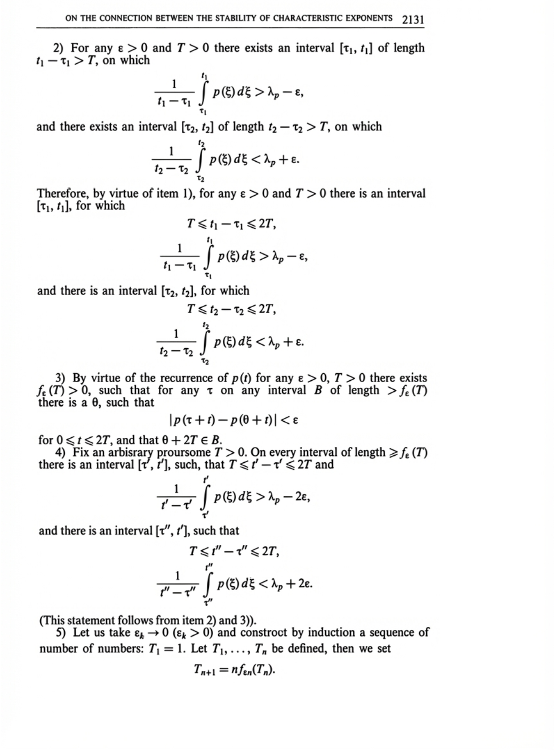
b) $u(t) = \ldots$ — is a continuous function, and $u(b) < \mu$ and $u(c) >$ of (8)). (8)

Case a). $u(t) = \ldots$ — is a continuous function, and $u(b) > \mu$ (by virtue of (6)), and $u(a) \leqslant \mu$ (by virtue of (7)). Therefore, there exists $t \in [a, b]$, such that $u(t) = \mu$; then the interval $[\rho_1, \rho_2] = [t, c]$ — is the required one.

Case b). $u(t) = \ldots$ — is a continuous function, and $u(b) < \mu$ (by virtue of (5)), and $u(c) > \mu$ (by virtue of (8)). Therefore, there exists $t \in [b, c]$, such that $u(t) = \mu$; then the interval $[\rho_1, \rho_2] = [a, t]$ — is the required one.

Case c). In this case $\ldots > \mu$, therefore it is considered in the same way as case b).

Figure 4: Figure 4



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2) For any $\varepsilon > 0$ and $T > 0$ there exists an interval $[\tau_1, t_1]$ of length $t_1 - \tau_1 > T$, on which

$$\frac{1}{t_1 - \tau_1} \int_{\tau_1}^{t_1} p(\xi)\, d\xi > \lambda_p - \varepsilon,$$

and there exists an interval $[\tau_2, t_2]$ of length $t_2 - \tau_2 > T$, on which

$$\frac{1}{t_2 - \tau_2} \int_{\tau_2}^{t_2} p(\xi)\, d\xi < \lambda_p + \varepsilon.$$

Therefore, by virtue of item 1), for any $\varepsilon > 0$ and $T > 0$ there is an interval $[\tau_1, t_1]$, for which

$$T \leqslant t_1 - \tau_1 \leqslant 2T,$$

$$\frac{1}{t_1 - \tau_1} \int_{\tau_1}^{t_1} p(\xi)\, d\xi > \lambda_p - \varepsilon,$$

and there is an interval $[\tau_2, t_2]$, for which

$$T \leqslant t_2 - \tau_2 \leqslant 2T,$$

$$\frac{1}{t_2 - \tau_2} \int_{\tau_2}^{t_2} p(\xi)\, d\xi < \lambda_p + \varepsilon.$$

3) By virtue of the recurrence of $p(t)$ for any $\varepsilon > 0$, $T > 0$ there exists $f_\varepsilon(T) > 0$, such that for any τ on any interval B of length $> f_\varepsilon(T)$ there is a θ, such that

$$|p(\tau + t) - p(\theta + t)| < \varepsilon$$

for $0 \leqslant t \leqslant 2T$, and that $\theta + 2T \in B$.

4) Fix an arbisrary proursome $T > 0$. On every interval of length $\geqslant f_\varepsilon(T)$ there is an interval $[\tau', t']$, such, that $T \leqslant t' - \tau' \leqslant 2T$ and

$$\frac{1}{t' - \tau'} \int_{\tau'}^{t'} p(\xi)\, d\xi > \lambda_p - 2\varepsilon,$$

and there is an interval $[\tau'', t']$, such that

$$T \leqslant t'' - \tau'' \leqslant 2T,$$

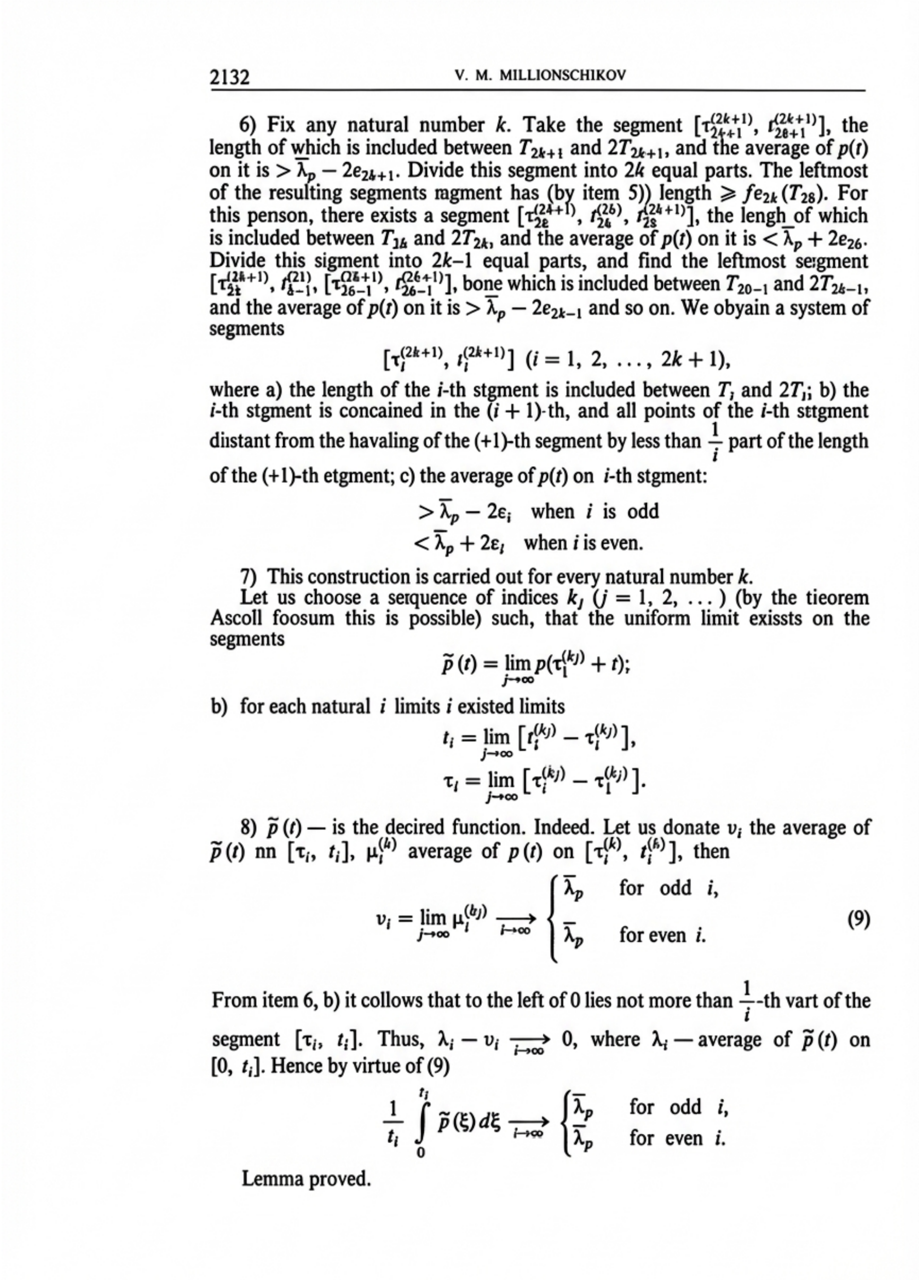
$$\frac{1}{t'' - \tau''} \int_{\tau''}^{t''} p(\xi)\, d\xi < \lambda_p + 2\varepsilon.$$

(This statement follows from item 2) and 3)).

5) Let us take $\varepsilon_k \to 0$ $(\varepsilon_k > 0)$ and constroct by induction a sequence of number of numbers: $T_1 = 1$. Let $T_1, \ldots, T_n$ be defined, then we set

$$T_{n+1} = n f_{\varepsilon n}(T_n).$$

Figure 5: Figure 5



6) Fix any natural number k. Take the segment $[\tau_{2k+1}^{(2k+1)}, t_{28+1}^{(2k+1)}]$, the length of which is included between T_{2k+1} and $2T_{2k+1}$, and the average of $p(t)$ on it is $> \overline{\lambda}_p - 2e_{2k+1}$. Divide this segment into $2k$ equal parts. The leftmost of the resulting segments magment has (by item 5)) length $\geqslant fe_{2k}(T_{28})$. For this penson, there exists a segment $[\tau_{2k}^{(2k+1)}, t_{2k}^{(26)}, t_{28}^{(2k+1)}]$, the lengh of which is included between T_{1k} and $2T_{2k}$, and the average of $p(t)$ on it is $< \overline{\lambda}_p + 2e_{26}$. Divide this sigment into $2k-1$ equal parts, and find the leftmost segment $[\tau_{2k}^{(2k+1)}, t_{k-1}^{(21)}, [\tau_{26-1}^{(2k+1)}, t_{26-1}^{(26+1)}]$, bone which is included between T_{20-1} and $2T_{2k-1}$, and the average of $p(t)$ on it is $> \overline{\lambda}_p - 2e_{2k-1}$ and so on. We obyain a system of segments

$$[\tau_i^{(2k+1)}, t_i^{(2k+1)}] \ (i = 1,\ 2,\ \ldots,\ 2k+1),$$

where a) the length of the i-th stgment is included between T_i and $2T_i$; b) the i-th stgment is concained in the $(i+1)$-th, and all points of the i-th stgment diistant from the havaling of the (+1)-th segment by less than $\frac{1}{i}$ part of the length of the (+1)-th etgment; c) the average of $p(t)$ on i-th stgment:

$$\begin{aligned} &> \overline{\lambda}_p - 2\varepsilon_i \quad \text{when } i \text{ is odd} \\ &< \overline{\lambda}_p + 2\varepsilon_i \quad \text{when } i \text{ is even.} \end{aligned}$$

7) This construction is carried out for every natural number k.

Let us choose a serquence of indices k_j $(j = 1,\ 2,\ \ldots)$ (by the tieorem Ascoll foosum this is possible) such, that the uniform limit exissts on the segments

$$\widetilde{p}(t) = \lim_{j\to\infty} p(\tau_1^{(k_j)} + t);$$

b) for each natural i limits i existed limits

$$t_i = \lim_{j\to\infty} \left[t_i^{(k_j)} - \tau_1^{(k_j)}\right],$$

$$\tau_i = \lim_{j\to\infty} \left[\tau_i^{(k_j)} - \tau_1^{(k_j)}\right].$$

8) $\widetilde{p}(t)$ — is the decired function. Indeed. Let us donate v_i the average of $\widetilde{p}(t)$ nn $[\tau_i,\ t_i]$, $\mu_i^{(k)}$ average of $p(t)$ on $[\tau_i^{(k)},\ t_i^{(k)}]$, then

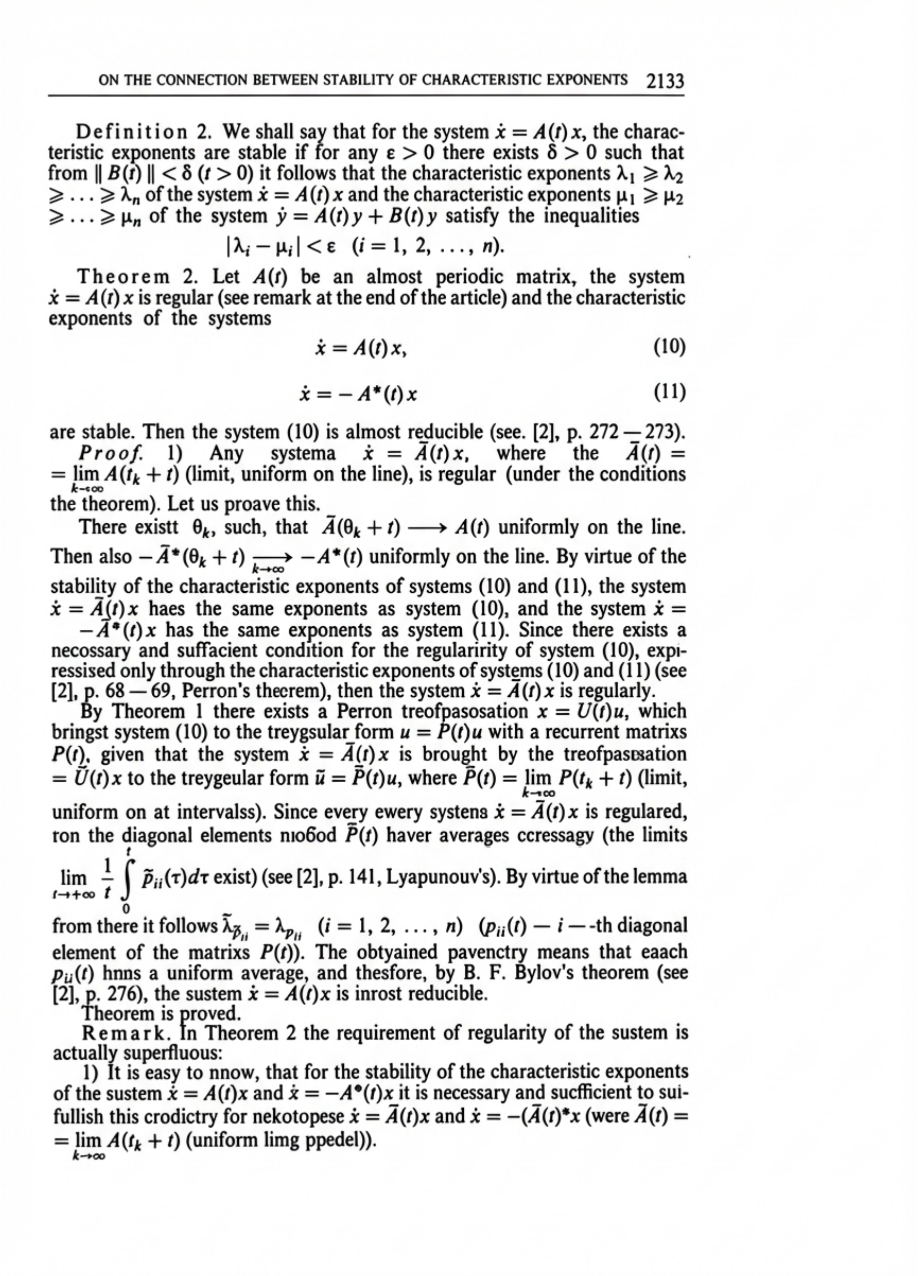
$$v_i = \lim_{j\to\infty} \mu_i^{(k_j)} \xrightarrow[i\to\infty]{} \begin{cases} \overline{\lambda}_p & \text{for odd } i, \\ \overline{\lambda}_p & \text{for even } i. \end{cases} \tag{9}$$

From item 6, b) it collows that to the left of 0 lies not more than $\frac{1}{i}$-th vart of the segment $[\tau_i,\ t_i]$. Thus, $\lambda_i - v_i \xrightarrow[i\to\infty]{} 0$, where λ_i — average of $\widetilde{p}(t)$ on $[0,\ t_i]$. Hence by virtue of (9)

$$\frac{1}{t_i}\int_0^{t_i} \widetilde{p}(\xi)\,d\xi \xrightarrow[i\to\infty]{} \begin{cases} \overline{\lambda}_p & \text{for odd } i, \\ \overline{\lambda}_p & \text{for even } i. \end{cases}$$

Lemma proved.

Figure 6: Figure 6



Definition 2. We shall say that for the system $\dot{x} = A(t)x$, the characteristic exponents are stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that from $\| B(t) \| < \delta$ $(t > 0)$ it follows that the characteristic exponents $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ of the system $\dot{x} = A(t)x$ and the characteristic exponents $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n$ of the system $\dot{y} = A(t)y + B(t)y$ satisfy the inequalities

$$|\lambda_i - \mu_i| < \varepsilon \quad (i = 1, 2, \ldots, n).$$

Theorem 2. Let $A(t)$ be an almost periodic matrix, the system $\dot{x} = A(t)x$ is regular (see remark at the end of the article) and the characteristic exponents of the systems

$$\dot{x} = A(t)x, \tag{10}$$

$$\dot{x} = -A^*(t)x \tag{11}$$

are stable. Then the system (10) is almost reducible (see. [2], p. 272 — 273).

Proof. 1) Any systema $\dot{x} = \bar{A}(t)x$, where the $\bar{A}(t) = \lim\limits_{k\to\infty} A(t_k + t)$ (limit, uniform on the line), is regular (under the conditions the theorem). Let us proave this.

There existt θ_k, such, that $\bar{A}(\theta_k + t) \longrightarrow A(t)$ uniformly on the line. Then also $-\bar{A}^*(\theta_k + t) \xrightarrow[k\to\infty]{} -A^*(t)$ uniformly on the line. By virtue of the stability of the characteristic exponents of systems (10) and (11), the system $\dot{x} = \bar{A}(t)x$ haes the same exponents as system (10), and the system $\dot{x} = -\bar{A}^*(t)x$ has the same exponents as system (11). Since there exists a necossary and suffacient condition for the regularirity of system (10), expıressised only through the characteristic exponents of systems (10) and (11) (see [2], p. 68 — 69, Perron's thecrem), then the system $\dot{x} = \bar{A}(t)x$ is regularly.

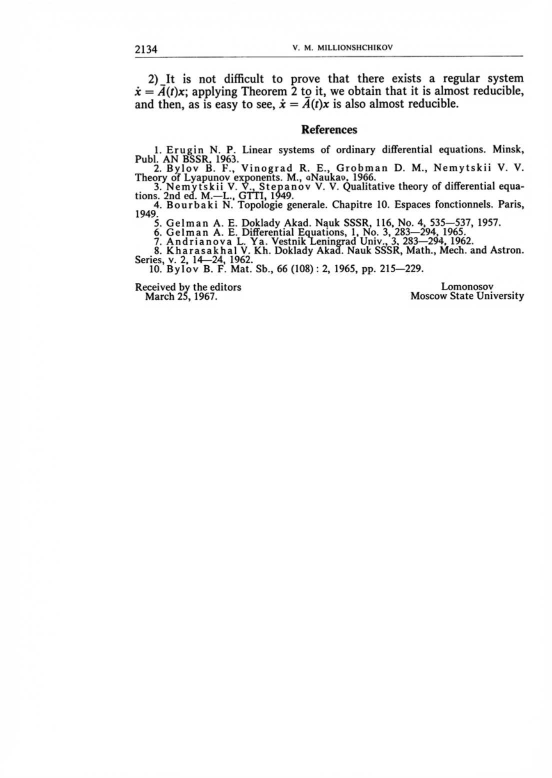
By Theorem 1 there exists a Perron treofpasosation $x = U(t)u$, which bringst system (10) to the treygsular form $u = P(t)u$ with a recurrent matrixs $P(t)$, given that the system $\dot{x} = \bar{A}(t)x$ is brought by the treofpassation $= \tilde{U}(t)x$ to the treygeular form $\tilde{u} = \bar{P}(t)u$, where $\bar{P}(t) = \lim\limits_{k\to\infty} P(t_k + t)$ (limit, uniform on at intervalss). Since every ewery systens $\dot{x} = \bar{A}(t)x$ is regulared, ron the diagonal elements nıoбod $\bar{P}(t)$ haver averages ccressagy (the limits $\lim\limits_{t\to+\infty} \frac{1}{t}\int\limits_0^t \tilde{p}_{ii}(\tau)d\tau$ exist) (see [2], p. 141, Lyapunouv's). By virtue of the lemma from there it follows $\tilde{\lambda}_{\tilde{p}_{ii}} = \lambda_{p_{ii}}$ $(i = 1, 2, \ldots, n)$ $(p_{ii}(t) - i$ —-th diagonal element of the matrixs $P(t)$). The obtyained pavenctry means that eaach $p_{ii}(t)$ hnns a uniform average, and thesfore, by B. F. Bylov's theorem (see [2], p. 276), the sustem $\dot{x} = A(t)x$ is inrost reducible.

Theorem is proved.

Remark. In Theorem 2 the requirement of regularity of the sustem is actually superfluous:

1) It is easy to nnow, that for the stability of the characteristic exponents of the sustem $\dot{x} = A(t)x$ and $\dot{x} = -A^*(t)x$ it is necessary and sucfficient to suifullish this crodictry for nekotopese $\dot{x} = \bar{A}(t)x$ and $\dot{x} = -(\bar{A}(t)^*x$ (were $\bar{A}(t) = \lim\limits_{k\to\infty} A(t_k + t)$ (uniform limg ppedel)).

Figure 7: Figure 7



2) It is not difficult to prove that there exists a regular system $\dot{x} = \bar{A}(t)x$; applying Theorem 2 to it, we obtain that it is almost reducible, and then, as is easy to see, $\dot{x} = \bar{A}(t)x$ is also almost reducible.

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Figure 8: Figure 8