INVESTIGATION OF THE DEFECT OF AN ABSTRACT CAUCHY PROBLEM

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Abstract

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MATHEMATICS

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INVESTIGATION OF THE DEFECT OF AN ABSTRACT CAUCHY PROBLEM

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Consider the abstract Cauchy problem

$$dx(t)/dt = Ax(t) \quad (0 < t < \infty), \qquad x(0) = x_0, \tag{1}$$

where A is a closed linear operator in a Banach space $\mathfrak{B}$. The set of those vectors x_0 for which problem (1) has a solution* (strongly continuously differentiable for $t > 0$ and strongly continuous for $t = 0$) is called the initial manifold and is denoted by I_A. In [1] some criteria are given for the density of the manifold I_A in the whole space under the condition that the domain D_A of the operator A is dense. If D_A is not dense, then the manifold I_A is all the more not dense, as is shown by the following simple

Lemma 1. *The inclusion always holds** *

$$\overline{I}_A \subseteq \overline{D}_A. \tag{2}$$

Indeed, if $x_0 \in I_A$, then $x_0 = \lim_{t\to 0} x(t)$, where $x(t)$ is the corresponding solution. But $x(t) \in D_A$ $(t > 0)$.

Relation (2) gives reason to introduce the following general

Definition. The dimension of the quotient space $\overline{D}_A/\overline{I}_A$ is called the defect of the Cauchy problem (1) and is denoted by d_A.

Now the subject of the present article is clear from its title.

Recall that, for any subspace $\mathcal{L}$, the dimension of the quotient space $\mathfrak{B}/\mathcal{L}$ is called the defect of the subspace $\mathcal{L}$ and is denoted by $\operatorname{def}\mathcal{L}$.

Lemma 2. *If the resolvent set of the operator A is nonempty, then the inequality holds*

$$\dim\left(\overline{D}_{A^k}/\overline{D}_{A^{k+1}}\right) \leq \operatorname{def}\overline{D}_A \quad (k = 1, 2, \ldots). \tag{3}$$

Proof. If $\{f_i\}_1^p$ is a linearly independent system of linear functionals on $\overline{D}_{A^k}$, orthogonal to $\overline{D}_{A^{k+1}}$, then $\{(R_\mu^k)^* f_i\}_1^p$ is a linearly independent system of linear functionals on all of $\mathfrak{B}$, orthogonal to $\overline{D}_A$. Here R_μ denotes, as usual, the resolvent of the operator A, i.e. $R_\mu = (A - \mu E)^{-1}$; μ is some point of the resolvent set.

We can now establish the following central result:

Theorem 1. *Let some half-plane* $\operatorname{Re}\lambda \geq a$ *belong to the resolvent set of the operator* A*, and in this half-plane*

$$\|R_\lambda\| = O(1 + |\lambda|^m) \tag{4}$$

for some $m \geq 0$*. Then***

$$d_A \leq [m+1] \operatorname{def} \overline{D}_A. \tag{5}$$

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* Not necessarily unique.
** The bar denotes closure.
*** [ ] denotes the integer part.

This theorem is, in essence, a generalization of Theorem 2 ([1]) and is proved in a similar way.*

Proof. Let $\operatorname{Re}\mu > \alpha$. Take an arbitrary vector x and put

$$u(t) = -\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{\lambda t} R_\lambda x}{(\lambda - \mu)^{[m+2]}}\, d\lambda \qquad (t \geqslant 0).$$

The vector-function $u(t)$ is strongly continuous. Put

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(\tau)\, d\tau = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{\lambda h} - 1}{\lambda} \frac{e^{\lambda t} R_\lambda x}{(\lambda - \mu)^{[m+2]}}\, d\lambda \qquad (h > 0).$$

The vector-function $u_h(t)$, for every $h > 0$, is a solution of problem (1) with $x_0 = u_h(0)$. Therefore $u_h(0) \in I_A$. Consequently, $u(0) \in \overline{I}_A$. But $u(0) = R_\mu^{[m+2]} x$. In view of the arbitrariness of the vector x,

$$\overline{D}_{A^{[m+2]}} \subseteq \overline{I}_A.$$

Hence, by Lemma 2,

$$d_A \leqslant \dim(\overline{D}_A/\overline{D}_{A^{[m+2]}}) = \sum_{k=1}^{[m+1]} \dim(\overline{D}_{A^k}/\overline{D}_{A^{k+1}}) \leqslant [m+1] \operatorname{def} \overline{D}_A,$$

as was required.

Theorem 1 is sharp first of all in the sense that the following is true.

Theorem 2. *For any integers $m \geqslant 0$, $p > 0$, there exists an operator A satisfying the conditions of Theorem 1 such that*

$$\operatorname{def} \overline{D}_A = p, \qquad d_A = (m+1)p. \tag{6}$$

Moreover, the following is true.

Theorem 3. *Let $\rho(r) > 1$ $(r \geqslant 0)$ be an arbitrary function growing faster than any power in the sense that $r^{-m}\rho(r) \to \infty$ $(r \to \infty)$, whatever $m > 0$. Then there exists an operator A, having no spectrum in the half-plane $\operatorname{Re}\lambda \geqslant 0$, such that*

$$\|R_\lambda\| \leqslant \rho(|\lambda|) \qquad (\operatorname{Re}\lambda \geqslant 0) \tag{7}$$

and at the same time

$$\operatorname{def} \overline{D}_A = 1, \qquad d_A = \infty.$$

The proof of Theorem 2 is reduced to the case $p = 1$ with the aid of the following simple proposition.

Lemma 3. *Let closed linear operators A_1, A_2 act in Banach spaces $\mathfrak{B}_1, \mathfrak{B}_2$, respectively. Form the direct sums $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, $A = A_1 + A_2$. Then*

$$\operatorname{def} \overline{D}_A = \operatorname{def} \overline{D}_{A_1} + \operatorname{def} \overline{D}_{A_2}, \qquad d_A = d_{A_1} + d_{A_2}. \tag{8}$$

Proof of Theorem 2 for the case $p = 1$. Let $\mathfrak{B} = C^{m+1}[0,1]$—the space of functions $\varphi(s)$ $(0 \leqslant s \leqslant 1)$ having a continuous $(m+1)$-st derivative;

$$\|\varphi\| = \max_{0 \leqslant n \leqslant m+1} \left(\max_{0 \leqslant s \leqslant 1} |\varphi^{(n)}(s)| \right).$$

Let A be the differentiation operator $-d/ds$ on functions having a continuous $(m+2)$-nd derivative and satisfying the boundary condition $\varphi(0) = 0$. Obviously, $\operatorname{def} \overline{D}_A = 1$.

The operator A has no spectrum: for any $\psi \in \mathfrak{B}$ and any λ,

$$R_\lambda \psi(s) = -e^{-\lambda s} \int_0^s \psi(\tau) e^{\lambda \tau} \, d\tau \qquad (0 \leqslant s \leqslant 1).$$

* Theorem 2 ([1]) corresponds to the case $\operatorname{def} \overline{D}_A = 0$. Then $d_A = 0$.

Hence, by integration by parts, one obtains the estimate

$$\max_{0 \le s \le 1} |R_\lambda \psi(s)| \le \frac{1}{|\lambda|} \left(2 \max_{0 \le s \le 1} |\psi(s)| + \max_{0 \le s \le 1} |\psi'(s)| \right) \quad (\operatorname{Re} \lambda > 0). \qquad (9)$$

But, since

$$(R_\lambda \psi(s))^n = -\lambda (R_\lambda \psi(s))^{(n-1)} - \psi^{(n-1)}(s)$$

$$(n = 1, 2, \dots, m+1;\ 0 \le s \le 1),$$

it follows, in view of (9), that

$$\max_{0 \le s \le 1} |(R_\lambda \psi(s))^{(n)}| = O\left(\|\psi\| (1 + |\lambda|^{n-1}) \right)$$

$$(n = 0, 1, \dots, m+1;\ \operatorname{Re} \lambda > 0),$$

i.e. condition (4) is satisfied in the half-plane $\operatorname{Re} \lambda > 0$.

At the same time, the solution of problem (1) with initial function $x_0 = \xi(s)$ has the form

$$x(t) = \xi(s - t) \max(0, \operatorname{sign}(s - t)).$$

In order that $\xi(s) \in L_A$, it is necessary and sufficient that

$$\xi^{(n)}(0) = 0 \quad (n = 0, 1, \dots, m+2).$$

Therefore

$$\Gamma_A = \{\xi \mid \xi^{(n)}(0) = 0 \ (n = 0, 1, \dots, m+1)\},$$

whence it is clear that $d_A = m + 1$.

We now prove Theorem 3. Let $\{M_n\}_0^\infty$ be a sequence of positive numbers $(M_0 = 1,\ M_1 \ge 1)$ satisfying the conditions

$$\sum_{k=0}^{\infty} \frac{r^k}{M_k} \le \rho(r) \quad (r \ge 0); \qquad M_{p+q} \ge M_{pM} q \quad (p, q = 0, 1, 2, \dots). \qquad (10)$$

It is easy to construct it inductively. Consider the Banach space $C\{M_n\}$ of infinitely differentiable functions $\varphi(s)$ $(0 \le s \le 1)$ with norm

$$\|\varphi\| = \sup_n \frac{1}{M_n} \max_{0\le s\le 1} |\varphi^{(n)}(s)|.$$

In this space, take the operator $\widetilde{A} = -d/ds$ on those functions $\varphi(s)$ for which $\varphi'(s) \in C\{M_n\}$, $\varphi(0) = 0$.

Let $\psi(s) \in C\{M_n\}$. Then the equation

$$-\varphi'(s) - \lambda\varphi(s) = \psi(s) \quad (0 \le s \le 1)$$

under the condition $\varphi(0) = 0$ has the unique solution

$$\varphi(s) = -e^{-\lambda s} \int_0^s \psi(\tau) e^{\lambda\tau}\, d\tau. \tag{11}$$

This solution belongs to $C\{M_n\}$ in the half-plane $\operatorname{Re}\lambda \ge 0$, since, recursively,

$$\max_{0\le s\le 1} |\varphi^{(n)}(s)| \le |\lambda|^n \max_{0\le s\le 1} |\psi(s)| + \sum_{k=1}^{n} |\lambda|^{n-k} \max_{0\le s\le 1} |\psi^{(k-1)}(s)| \quad (n = 0, 1, 2, \ldots),$$

whence, by (10),

$$\|\varphi\| \le \|\psi\| \sum_{k=0}^{n} \frac{|\lambda|^k}{M_k} \le \rho(|\lambda|)\|\psi\|.$$

We see that the operator $\widetilde{A}$ has no spectrum in the half-plane $\operatorname{Re}\lambda \ge 0$, and its resolvent satisfies inequality (7). When is $\operatorname{def}\overline{D}_{\widetilde{A}} = 1$? This question appears difficult (see, for example, (2), pp. 240–243). We shall avoid it by considering, instead of the operator $\widetilde{A}$, a certain restriction $\bar{A}$. Let Π be the linear manifold of all polynomials. Obviously, $\Pi \subset C\{M_n\}$. Put

$$\Pi_0 = \Pi \cap \{\varphi \mid \varphi(0) = 0\}.$$

The operator $\widetilde{A}$ everywhere on Π_0

is defined and maps Π_0 into Π. Restrict it to Π_0 and close it. This gives a certain operator $A \subset \widetilde{A}$, acting in the Banach space $\mathfrak{B} = \overline{\Pi} \subset C\{M_n\}$. Obviously, $\overline{D}_A = \overline{\Pi}_0$ and, therefore, $\operatorname{def}\overline{D}_A = 1$. Further, if $\xi(s) \subset I_A$, then $\xi^{(n)}(0) = 0$ $(n = 0, 1, 2, \ldots)$ and, consequently, $d_A = \infty$. It remains to verify that, when the operator $\widetilde{A}$ is restricted to the operator A, no residual spectrum appears. For this it is enough to prove that if in (11) the function ψ is a polynomial, then $\varphi \subset \Pi$. This assertion, in turn, reduces to the inclusions

$s^k e^{-\lambda s} \subset \overline{\Pi}$ ($k = 0, 1, 2, \dots;\ \operatorname{Re}\lambda \geqslant 0$). But the point is that the expansion of $e^{-\lambda s}$ in a Taylor series, as is not difficult to verify, converges in the space $C\{M_n\}$, and the expansions of the remaining functions $s^k e^{-\lambda s}$ are obtained by repeated differentiation with respect to λ. Thus Theorem 3 is proved.

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References

1. Yu. I. Lyubich, DAN, 155, No. 2 (1964).
2. S. Mandelbrojt, *Adherent Series*, IL, 1955.

Note: Figure translations are in progress. See original paper for figures.