THE METHOD OF PROJECTIONS IN THE STUDY OF SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract

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THE METHOD OF PROJECTIONS IN THE STUDY OF SOLUTIONS OF ELLIPTIC EQUATIONS

1. We shall consider functions in a bounded domain G of the n-dimensional Euclidean space E^n. Let $E^m = E$ be some m-dimensional plane, $1 \leq m \leq n$; x_E is the projection of the point x, and G_E is the projection of G. We define the (lower) projection u_E of the function u as such a function in G_E that

$$u_E(x') = \inf_{x_E = x'} u(x), \qquad x' \in G_E, \quad x \in G. \tag{1}$$

One may regard u_E as defined on G, putting $u_E(x) = u_E(x_E)$. Then, if at some point $u(x) = u_E(x)$ and du, du_E, d^2u, d^2u_E exist, then at such a point

$$u = u_E, \qquad du = du_E, \qquad d^2u \geq d^2u_E. \tag{2}$$

Let $F(u_{ij}, u_i, u, x)$ be some elliptic expression, so that the form $F_{u_{ij}}\xi^i\xi^j$ is positive. Then, if $v_{ij}\xi^i\xi^j \geq 0$, then

$$F'_t(u_{ij} + tv_{ij}, u_i, u, x)_{t=0} = F_{u_{ij}}v_{ij} \geq 0, \tag{3}$$

i.e. F increases with the growth of the form $u_{ij}\xi^i\xi^j$. Therefore, at a point x where (2) is fulfilled, we shall have

$$F(u_{ij}, u_i, u, x) \geq F(u_{Eij}, u_{Ei}, u_E, x),$$

and, taking the projection of F itself, all the more we obtain

$$F(u_{ij}, u_i, u, x) \geq F_E(u_{Eij}, u_{Ei}, u_E x_E). \tag{4}$$

Now we may regard u_E as a function in G_E and have in view its derivatives with respect to the coordinates in the plane E (by rotating the axes in advance, one may make E the plane of $x^1, \dots, x^m$). Then it is obvious that the right-hand side of (4) pertains to the function u_E only of m variables. If the original function u satisfies the inequality $F \le 0$, then u_E satisfies the corresponding inequality $F_E \le 0$ in the projections x' of points x where $u(x) = u_E(x')$ and u, u_E are twice differentiable. Of course, this will not be so for all $x' \in G_E$. For example, if u is continuous in $G + \Gamma$ (Γ is the boundary of G), $u \le u_0 = \text{const}$, $u|_\Gamma = u_0$, then for every $x' \in G_E$ there is certainly a point $x \in G$ with $x_E = x'$, $u(x) = u_E(x')$. But if, moreover, u is twice continuously differentiable, then u_E may turn out not to be differentiable; it will be twice differentiable almost everywhere. However, in some respects this proves to be immaterial. For example, from the inequality $F_E \le 0$, under quite general conditions, one can estimate u_E from below, and then, since $u(x) \ge u_E(x_E)$, one thereby obtains a lower estimate for u itself.

Defining the upper projection u^E: $u^E(x') = \sup_{x_E = x'} u(x)$, $x' \in G_E$, we obtain, in a completely analogous way to (4),

$$F(u_{ij}, \dots, x) \le F^E(u^E_{ij}, \dots, x_E).$$

Therefore, if u is a solution of the elliptic equation $F = 0$, then for u_E and u^E we obtain the inequalities $F_E \le 0$, $F^E \ge 0$. Since the upper and lower projections interchange roles when the sign of the function is changed, it will be sufficient to consider the lower projection.

2.

We now formulate the precise basis of the outlined "method of projections" under the assumptions adopted in ([1],[2]) in estimating solutions of elliptic equations.

Let the function u be bounded and lower semicontinuous and have an absolutely continuous support mapping. Let $u|_\Gamma \ge 0$, defining, for $x \in \Gamma$,

$$u(x) = \liminf_{x' \to x} u(x'), \qquad x' \in G.$$

(For arbitrary $u|_\Gamma$ one may introduce a convex function v with $v|_\Gamma \le |u|_\Gamma$; then $u' = u - v$, as is easy to see ([1]), also has an absolutely continuous support mapping and $u'|_\Gamma \ge 0$.) When $u|_\Gamma \ge 0$, it is enough to assume that the negative part u^- of the function u has an absolutely continuous support mapping.

Suppose, further, that at almost all points of convexity at which $u < 0$ and $\det(u_{ij}) > 0$, the function u satisfies the inequality

$$F(u_{ij}, u_i, u, x) \le 0, \tag{5}$$

where F is a nondecreasing function of the matrix u_{ij} for $(u_{ij}) \ge 0$ and arbitrary given u_i, $u < 0$, $x \in G$. The notation $(u_{ij}) \ge 0$ means that $u_{ij}\xi^i\xi^j \ge 0$, and the monotonicity of F is understood correspondingly: $(u_{ij}) \geq (u'_{ij})$ if $(u_{ij} - u'_{ij}) \geq 0$. This monotonicity requirement on F, as is clear from Sec. 1, is a natural generalization of ellipticity. (We also note that, as shown in (^{1}), any function has approximate differentials du, d^2u at almost all points of convexity. Therefore, understanding the derivatives u_i, u_{ij} as the coefficients of these differentials, there is no need for special conditions for their existence. They are equivalent to ordinary or generalized derivatives when such derivatives exist.)

Theorem 2. *Under the formulated assumptions on $u(x)$, for any m $(1 \leq m \leq n)$, for almost all planes $E^m = E$ of any bundle, the projections u_E satisfy analogous conditions: u_E^- has an absolutely continuous support mapping, and at almost all points of convexity $x \in G_E$, where $u_E < 0$ and $\det(u_{Eij}) > 0$,*

$$F_E(u_{Eij}, u_{Ei}, u_E, x) \leq 0. \tag{6}$$

Here, obviously, F_E will also be a nondecreasing function of (u_{Eij}) for $(u_{Eij}) \geq 0$, and $u_E \geq 0$ on the boundary of G_E. Thus the theorem, briefly speaking, consists in the fact that all the conditions formulated above are preserved when projecting u and F onto almost every plane of any bundle.

The proof is obtained by applying the elementary considerations of Sec. 1 to the points of convexity of the function u_E^- and using Lemma 1 of (^{1}). If $x' \in G_E$ is a point of convexity of u_E^-, then, obviously, there exists a point $x \in G$ such that $x_E = x'$, $u(x) = u_E(x')$, and it will be a point of convexity of the function u. The approximate differentials $du, \dots, d^2u_E$ exist at almost all points of convexity, and $d^2u \geq 0$, $d^2u_E \geq 0$.

The conclusions of the papers (^{1},^{2}), where projection onto planes is used, may be regarded as applications of Theorem 1.

3.

The simplest case is projection onto lines. Then (6) reduces to an inequality in ordinary derivatives. The line E may be taken as the axis x^1. Then, if F is such that for sufficiently large p

$$F(u_{11} + p, u_{12}, \dots, u_i, u, x) > 0,$$

then inequality (5) is solvable with respect to u_{11}:

$$u_{11} + H(u_{12}, \dots, x) \leq 0.$$

Projecting onto E and putting

$$H_E(0, \dots, 0, u_1, 0, \dots, 0, u, x_E) = K(u_1, u, x^1),$$

we obtain

$$u_E'' \le K(u_E', u_E, x), \qquad x = x^1. \tag{7}$$

This inequality is valid at almost all points of convexity of the function u_E, at which $u_E < 0$, $u_E'' > 0$. If v is a convex function stretched over u_E^-, then $v'' = 0$ everywhere where $v \ne u_E$, and $v'' \le u_E''$ (and even $v'' = u_E''$) at almost all points at which $v = u_E$. Therefore from (7) it follows that almost everywhere on G_E

$$v'' \le K_+(v', v, x), \tag{8}$$

where K_+ is the positive part of K. Moreover, at the endpoints of the segment G_x, $v = 0$, and the absolute continuity of the supporting image of the function u_x^- is, obviously, equivalent to the same property of v, i.e., to the absolute continuity of its derivative v'.

If we want to estimate the original function u from below, it suffices to estimate v, and since v is convex and at the endpoints $v = 0$, for this it is enough to estimate v' at the endpoints, and even only at one of them. This is reduced in an obvious way to estimating the solution of the equation $y' = K_+(y, v, x)$, if the conditions of existence and uniqueness are satisfied for it. Namely, if $v'(x_0) = 0$ and $y(x)$ is a solution with $y(x_1) = 0$, where $x_1 \le x_0$, then for $x \ge x_1$, $y(x) \ge v'(x)$. This also gives estimates of the normal derivative u_ν at points of convexity Γ at which $u = 0$.

4. Application of the method described above to the simplest case of a linear inequality gives the following results. Let

$$a^{ij}u_{ij} + b_i^{iu} + cu \le f, \qquad a^{ij}\xi_i\xi_j \ge 0. \tag{9}$$

Let E be a straight line. Define for a function φ in G the norm $\|\varphi\|_E$ in the same way as was done in (2). Namely, taking E as the x^1-axis, we consider measurable functions $\psi(x^1)$ on the segment G_E such that everywhere in G

$$|\varphi(x^1, \dots, x^n)|/a^{11}(x^1, \dots, x^n) \le \psi(x^1), \tag{10}$$

and set

$$\|\varphi\|_E = \inf \|\psi\|_{L_1(G_E)} \tag{11}$$

the infimum being over all ψ satisfying (10), not excluding a priori infinite values of the norms. For brevity we write $\|b^1\|_E = \|b\|_E$, $\|b_\pm\| = \|b_\pm\|_E$, where plus and minus denote the positive and negative parts of b^1.

If the origin has been moved to the midpoint of the segment G_E, then set $x^1 = l_E \xi$, where l_E is half the length of G_E.

Theorem 2. *If the function $u(x)$ satisfies the conditions of item 2 and the inequality (9), then at every point where $u(x) < 0$, for almost all straight lines E the inequality*

$$|u(x)| < \frac{\|(f - cu)_+\|_E \, e^{\|b\|_E} \, l_E (1 - \xi^2)}{e^{\|b_-\|_E}(1 - \xi) + e^{\|b_+\|_E}(1 + \xi)}, \tag{12}$$

holds, where the norms may be taken over that part $G(u < 0)$ of the domain G where $u < 0$.

Since $\|b\| = \|b_+\| + \|b_-\|$, by finding the minimum of the denominator as a function of $\|b_-\|$, $\|b_+\|$ under $\|b_-\| + \|b_+\| = \|b\|$, $\|b_-\|, \|b_+\| \geq 0$, we obtain

$$|u(x)| < \|(f - cu)_+\|_E \, l_E \begin{cases} \dfrac{1}{2} e^{\|b\|_E/2} \sqrt{1 - \xi^2}, & \xi \leq \operatorname{th} \dfrac{\|b\|_E}{2}, \\ \dfrac{e^{\|b\|_E}(1 - \xi^2)}{e^{\|b\|_E}(1 - |\xi|) + (1 + |\xi|)}, & \xi \leq \operatorname{th} \dfrac{\|b\|_E}{2}. \end{cases} \tag{13}$$

The right-hand sides of (12) and (13) are convex functions of ξ, and in (13) it is differentiable also at $\xi = \operatorname{th} \|b\|/2$.

Let us note that when $u < 0$, $(f - cu)_+ \leq f_+ + c_+|u|$, and, consequently, for the norms in $G(u < 0)$, $\|(f - cu)_+\| \leq \|f_+\| + \|c_+ u\|$. Therefore from (12) and (13) it follows that

$$|u(x)| < (\|f_+\|_E + \|c_+ u\|_E) M_E(x), \tag{14}$$

where M_E denotes the corresponding multiplier either from (12) or from (13), considered as a function of x. Hence, multiplying by c_+ and taking norms of both sides, we obtain

$$\|c_+ u\|_E < (\|f_+\|_E + \|c_+ u\|_E) \|c_+ M\|_E, \tag{15}$$

provided only that $\|c_+\| < \infty$. For $f_+ = 0$ this gives

$$\|c_+ M\|_E > 1. \tag{16}$$

This proves

Theorem 3. *In order that the homogeneous inequality (9) admit a solution with* $u|_\Gamma \geq 0$ *that takes negative values, it is necessary that, for almost all lines* E *for which* $\|c_+\|_E < \infty$*, the inequalities (16) be satisfied.*

If, however, $\|c_+M\|_E < 1$, then from (15) one can estimate $\|c_+u\|_E$, and, substituting this estimate into (14), we obtain

$$|u(x)| < \frac{\|f_+\|_E M_E(x)}{1 - \|c_+M\|_E}, \tag{17}$$

i.e., this proves

Theorem 4. *At all points where* $u < 0$*, for almost all lines* E *for which* $\|c_+\|_E < \infty$ *and* $\|c_+M\|_E < 1$*, the inequalities (17) hold.*

The estimates (12), (14), (17) naturally imply estimates of the normal derivative u_ν at points of convexity of Γ at which $u = 0$.

5. All the estimates obtained are sharp, as the following theorem shows.

Theorem 5. *Let* G *be a right cylinder formed by segments of length* $2l_E$*, parallel to the given line* E*, with an arbitrary* $(n-1)$*-dimensional domain as base. Let* x_0 *be a given point inside* G*. For each of the inequalities (14), (18), and for every* $\varepsilon > 0$*, one can construct in* G *a linear equation with a solution* $u(x)$*,* $u|_\Gamma = 0$*, such that* $-u(x_0)$ *differs from the right-hand side of the corresponding inequality for the given line* E *by less than* ε*. Moreover, the norms entering these inequalities may be assigned any prescribed nonnegative values, provided that* $\|b_-\| + \|b_+\| = \|b\|$*,* $\|f_+\| + \|c_+u\| > 0$*, and in (18)* $\|f_+\| > 0$*,* $\|c_+M\| < 1$*.*

In addition, in G *one can specify a homogeneous equation for which* $\|c_+M\|_E < 1+\varepsilon$*, but which has a nontrivial solution with* $u|_\Gamma = 0$*. (This shows the sharpness of condition (16).)*

In all cases it can be ensured that the equation is strictly elliptic and that its coefficients and right-hand side, as well as the solution, are differentiable arbitrarily many times inside G.

All these results substantially refine what was obtained in [2,3] for the same case, when E are straight lines.

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Note: Figure translations are in progress. See original paper for figures.