SIMPLE BASES IN CLOSED CLASSES OF FUNCTIONS OF THE ALGEBRA OF LOGIC

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.11

MATHEMATICS

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SIMPLE BASES IN CLOSED CLASSES OF FUNCTIONS OF THE ALGEBRA OF LOGIC

(Presented by Academician P. S. Novikov on 21 IX 1965)

A complete description of all closed classes of functions of the algebra of logic was given by Post ([1]); for each of these classes he found all its maximal proper subclasses, called *precomplete* classes.

Post established a criterion for completeness of a system of functions in a given closed class: a necessary and sufficient condition for completeness of a system of functions in a given closed class is the presence in this system of at least one function not belonging to each of the precomplete classes (for the given closed class). From the fact that the number of all precomplete classes in any closed class is finite (does not exceed 5), it follows that every closed class of functions of the algebra of logic has a finite basis (a minimal complete system) ([2,3]).

Following ([4]), we shall call a certain basis in a given closed class a *simple basis* if no function entering into it can be replaced, by identifying its variables, by one or several functions of a smaller number of variables in such a way that the system of functions obtained as a result of this replacement would remain a complete system. While the total number of bases in a given closed class is, generally speaking, infinite, the number of its simple bases is always finite ([5,6]).

In ([4]) a description was given of all simple bases in the class C_1—the class of all functions of the algebra of logic*. In the present paper a description is obtained of all simple bases in each of the closed classes of functions of the algebra of logic.

For each class all its simple bases are indicated, with the exception of four groups of classes

$$F_1^\mu,\ F_4^\mu,\ F_5^\mu,\ F_8^\mu,\ \mu \geqslant 2,$$

for which examples of such bases are given and an estimate for their total number is given. All the results directly rely on the results of ([3]), from which all notation is also taken.

We present some of the results obtained in the form of a table.

* Let us note that in ([4]) it is incorrectly stated that the total number of simple bases of the class C_1 is 48. In fact this number is 44, since the bases
(1) a) $x_1x_2 + x_1x_3 + x_2x_3 + 1$, b) $x_1x_2 + x_1x_3 + x_2x_3 + x_1 + x_2 + 1$;
2) a) x_1x_2, b) $x_1 \vee x_2$,
listed among the simple ones are not simple—they are reducible, by identifying variables, to bases consisting of conjunction (or disjunction) and negation.

Table 1

Closed class	Designation	Simple bases of the class
All functions of one variable	O_9	**1.** $\{(1)\ \bar{x};\quad 2)\ 1\}$. **2.** $\{(1)\ \bar{x};\quad 2)\ 0\}$.
Linear functions preserving 1	L_2	**1.** $\{x+y+1\}$. **2.** $\{(1)\ x+y+z;\quad 2)\ 1\}$.
Linear self-dual functions	L_5	**1.** $\{x+y+z+1\}$.**2.** $\{(1)\ x+y+z;\quad 2)\ \bar{x}\}$.
All linear functions	$L_1 = L$	**1,2.** $\{(1)\ x+y;\quad 2)$ a) $1,$ b) $\bar{x}\}$.**3,4.** $\{(1)\ x+y+1;\quad 2)$ a) $0,$ b) $\bar{x}\}$.**5,6.** $\{(1)\ x+y+z+1;\quad 2)$ a) $0,$ b) $1\}$.**7,8.** $\{(1)\ x+y+z;\quad 2)\ \bar{x};\quad 3)$ a) $0,$ b) $1\}$.**9.** $\{(1)\ x+y+z;\quad 2)\ 0;\quad 3)\ 1\}$.
Self-dual monotone functions	D_2	$\{xy \vee xz \vee yz\}$.
All self-dual functions	$D_3 = S$	**1.** $\{\bar{x}y \vee \overline{xz} \vee \overline{yz}\}$. **2.** $\{x\bar{y} \vee x\bar{z} \vee \overline{yz}\}$.**3.** $\{(1)\ x\bar{y} \vee xz \vee yz;\quad 2)\ \bar{x}\}$.**4.** $\{(1)\ xy \vee xz \vee yz;\quad 2)\ \bar{x}\}$.
Monotone functions, except functions equal to 0	A_2	**1,2** $\{(1)$ a) $xy \vee xz \vee yz,$b) $x \vee y;\quad 2)\ xy;\quad 3)\ 1\}$.**3.** $\{(1)\ x(y \vee z);\quad 2)\ 1\}$.
Monotone functions not equal to constants	A_4	$\{xy,\quad x \vee y\}$.
All monotone functions	$A_1 = M$	**1,2,3.** $\{(1)$ a) $xy \vee xz \vee yz,\quad$ b) $x(y \vee z),$c) $x \vee yz;\quad 2)\ 0;\quad 3)\ 1\}$.**4.** $\{(1)\ x \vee y;\quad 2)\ xy;\quad 3)\ 0;\quad 4)\ 1\}$.

Closed class	Designation	Simple bases of the class
Functions preserving 0 and 1	C_4	**1.** $\{xyz+xy+xz+yz+x\}$. **2.** $\{xyz+x+y\}$. **3.** $\{xyz+xy+xz+y+z\}$.**4.** $\{xy+xz+y\}$. **5.** $\{xyz+xy+x+y+z\}$. **6.** $\{xy+y+z\}$.**7–10.** $\{$(1) a) $xy+xz+yz+x+y$,b) $x+y+z$; 2) a) xy, b) $x\vee y\}$.**11, 12.** $\{$(1) a) $xyz+xy+x$,b) $xy+xz+x$; 2) $x\vee y\}$.**13,14.** $\{$(1) a) $xyz+xy+xz+x+y$,b) $xy+xz+x+y+z$; 2) $xy\}$.

(continued)

Closed class	Designation	Simple bases of the class
Functions preserving 1	C_2	**1.** $\{xyz+z+1\}$.**2.** $\{xyz+xy+xz+yz+1\}$.**3.** $\{xz+yz+1\}$.**4.** $\{xy+xz+yz+x+1\}$.**5.** $\{xy+xz+yz+x+y+z+1\}$.**6–14.** {1) a) $xyz+xy+xz+yz+x$,b) $xyz+x+y$,c) $xyz+xy+xz+y+z$,d) $xyz+xy+x+y+z$,e) $xy+xz+y$, f) $xy+y+z$, g) $xy+xz+yz+x+y$, h) $xyz+xy+x$,i) $xy+xz+x$; 2) 1}.**15–17.** {1) $x \vee y$; 2) a) $x+y+z$,b) $x+y+1$, c) $xy \vee xz \vee yz$}.**18–20.** {1) a) xy, b) $x \vee y$,c) $xy \vee xz \vee yz$; 2) $x+y+1$}.**21,22.** {1) a) $xyz+xy+xz+x+y$,b) $xy+xz+x+y+z$; 2) xy; 3) 1}.**23–25.** {1) a) xy, b) $x \vee y$,c) $xy \vee xz \vee yz$; 2) $x+y+z$; 3) 1}.
Monotone functions satisfying the condition $\langle a^\mu \rangle$, $\mu \geq 2$	F_3^μ	{1) $h_\mu^*(x_1 \dots x_\mu + 1)$; 2) 1}*.
Monotone functions satisfying the condition $\langle a^\infty \rangle$	F_3^∞	{1) $x \vee yz$; 2) 1}.
All functions satisfying the condition $\langle a^\infty \rangle$	F_4^∞	**1.** $\{x \vee \bar{y}\}$.**2,3** {1) a) $x \vee \bar{y}z$, b) $x \vee \bar{y}z \vee y\bar{z}$; 2) 1}.

Closed class	Designation	Simple bases of the class
All functions satisfying the condition $\langle a^\mu \rangle$, $\mu \geq 2$	F_4^μ	There exist 4 simple bases of order $\mu+1$:**1.** $\{h_\mu^*(x_1 \dots x_\mu+1)\}^{**}$.**2.** $\{1)\ h_\mu^*(x_1 \dots x_\mu + 1);\ 2)\ x \vee \bar{y}\}$.**3,4.** $\{1)\ h_\mu^*(x_1 \dots x_\mu+1);2)$ a) $xyz+xy+xz+x+y$,b) $xy+xz+x+y+z$; 3) $1\}$.The remaining simple bases may consist of no more than two functions; moreover, one of these functions has order n:$\mu+1 < n \leq 2^\mu+1-1$,and either it is the only function of the simple basis, or the second function of it is equal to 1.

$$^{*}\ h_\mu^*(x_1 \dots x_\mu+1) = \prod_{i=1}^{\mu+1}(x_1 \vee \dots \vee x_{i-1} \vee x_{i+1} \vee \dots \vee x_{\mu+1}).$$

$$^{**}\ h_\mu^*(x_1 \dots x_\mu+1) = \prod_{i=1}^{\mu+1}(x_1 \vee \dots \vee \bar{x}_i \vee \dots x_{\mu+1}).$$

The closed classes $L_3, A_3, C_3, F_1^\mu, F_7^\infty, F_8^\infty, F_8^\mu$, $\mu \geqslant 2$, are respectively dual to the classes $L_2, A_2, C_2, F_3^\mu, F_3^\infty, F_4^\infty, F_4^\mu$, $\mu \geqslant 2$. Their simple bases are dual to the simple bases of the classes dual to them.

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Received
17 IX 1965

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Note: Figure translations are in progress. See original paper for figures.