ON FREE POLYNILPOTENT GROUPS

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Abstract

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We shall use the following notation: G_s is the s-th term of the lower central series of the group G, defined inductively: $G_s = [G_{s-1}, G]$, if $s > 1$, and $G_1 = G$; if a sequence $n_1, n_2, \dots, n_k, \dots$ of positive integers is given, then $G_{n_1,\dots,n_k}$ denotes the subgroup, also defined inductively: $G_{n_1,\dots,n_k} = (G_{n_1,\dots,n_{k-1}})_{n_k}$, the n_k-th term of the lower central series of the group $G_{n_1,\dots,n_{k-1}}$, if $k > 1$.

If $G_{n_1,\dots,n_k} = E$, then the group G is called **polynilpotent** with respect to the sequence $n_1, \dots, n_k$; all such groups form the variety $\mathfrak{P}$ of polynilpotent groups with respect to the sequence $n_1, \dots, n_k$, which is represented in the form of the product

$$\mathfrak{P} = \mathfrak{N}_{n_k-1}\mathfrak{N}_{n_{k-1}-1}\cdots\mathfrak{N}_{n_1-1};$$

here $\mathfrak{N}_c$ denotes the variety of all nilpotent groups of class $\leq c$.

The free groups of the variety $\mathfrak{P}$ are called **free polynilpotent groups**. A free polynilpotent group G is represented in the form of a factor group $G \cong F/F_{n_1,\dots,n_k}$ of a free group F of the same rank as the group G. In what follows, by G we shall denote a free polynilpotent group of arbitrary rank with respect to the sequence $n_1, \dots, n_k$.

Free polynilpotent groups have been studied in a number of works (see ([1–5])). The present note is a natural continuation of the author' s investigations ([2,3]).

In ([2,3]) the W-free subgroups of free soluble groups were completely described, where W is an arbitrary variety (we recall that free soluble groups are a special case of free polynilpotent groups when $n_1 = \cdots = n_k = 2$). Here we formulate a theorem describing the W-free subgroups of free polynilpotent groups, from which the theorem on free soluble groups follows directly.

Theorem 1. Let H be a subgroup of the group G such that $H \subseteq G_{n_1,\dots,n_s}$, but $H \nsubseteq G_{n_1,\dots,n_{s+1}}$. If H is W-free with respect to some variety W, then H is a free polynilpotent group. The subgroup H is a free polynilpotent group if and only if it has a system of generators $h_i,\ i \in I$, which modulo $G_{n_1,\dots,n_{s+1}}$ freely generate a free nilpotent group. If the class of this free nilpotent group is equal to $c - 1$ ($c \leq n_{s+1}$), then H corresponds to the sequence $c, n_{s+2}, \dots, n_k$.

In the proof of this theorem, besides the known properties of free polynilpotent groups, the following lemmas are used.

Lemma 1. A W-free subgroup of a free nilpotent group is free nilpotent.

Lemma 2. Let F/N be a free nilpotent group of finite rank r, where F is a free group. Then in F one can choose a system of free generators $x_1, \dots, x_r, \dots$ such that the images of the elements $x_1, \dots, x_r$ in F/N are free generators of the group F/N.

The following theorems were also unknown for free solvable groups.

Theorem 2. Let H be such a free polynilpotent subgroup of the group G that

$$H \supseteq G_{n_1,\dots,n_{k-1}},$$

then H coincides with one of the subgroups $G_{n_1,\dots,n_s}$, $s = 1, \dots, k-1$.

Theorem 3. Let H be such a free polynilpotent subgroup of the group G that

$$H \supseteq G_{n_1,\dots,n_{k-1},\, n_k - 1};$$

then H either coincides with one of the subgroups $G_{n_1,\dots,n_s}$, $s = 1, \dots, k-1$, or H is abelian.

Finally, the following theorem completely describes the free polynilpotent normal divisors of the group G.

Theorem 4. Let H be an invariant free polynilpotent subgroup of the group G. Then either H coincides with one of the subgroups $G_{n_1,\dots,n_s}$, $s = 1, \dots, k-1$, or H is free abelian and lies in $G_{n_1,\dots,n_{k-1}}$; in the latter case, if $k \geq 2$, then

$$H \subseteq G_{n_1,\dots,n_{k-1},\, l},$$

where

$$l = \left[\frac{n_k + 1}{2}\right].$$

Corollary. Every free nilpotent normal divisor of a free nilpotent group either coincides with the whole group, or is abelian.

Let us also note that from the results of the work [6] it can be deduced that the set of nonisomorphic three-step solvable groups with two generators has the cardinality of the continuum. This shows that free polynilpotent groups are very rich in normal divisors.

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Note: Figure translations are in progress. See original paper for figures.