

ON SOME IDENTITIES OF THE GENERAL THEORY OF ORTHOGONAL POLYNOMIALS AND THEIR APPLICATIONS

MATHEMATICS

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.21788>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.512.6+517.512.7

MATHEMATICS

D. L. BERMAN

ON SOME IDENTITIES OF THE GENERAL THEORY OF ORTHOGONAL POLYNOMIALS AND THEIR APPLICATIONS

(Presented by Academician S. N. Bernstein, 4 I 1966)

1°. Let $\{\omega_k(x)\}_{k=0}^{\infty}$ be a system of polynomials orthonormal on the interval $[-1, 1]$ with respect to the weight $g(x)$, and let a_n be the leading coefficient of $\omega_n(x)$. It is known that the recurrence formula holds

$$\omega_k(x) = (\alpha_k x + \beta_k)\omega_{k-1}(x) - \gamma_k\omega_{k-2}(x), \quad k = 2, 3, \dots, \quad (1)$$

where $\alpha_k, \beta_k, \gamma_k$ are constants, with $\alpha_k = a_k/a_{k-1}$, $\gamma_k = \alpha_k/\alpha_{k-1}$.

Of importance in the general theory of orthogonal polynomials is the Christoffel-Darboux formula

$$\sum_{k=0}^n \omega_k(x)\omega_k(t) = \frac{a_n}{a_{n+1}} \frac{\omega_{n+1}(t)\omega_n(x) - \omega_n(t)\omega_{n+1}(x)}{t-x}. \quad (2)$$

In the present note we study the sum $\sum_{k=0}^n \omega'_k(x)\omega'_k(t)$, where the prime, as usual, denotes the first derivative. We shall show that this sum is also of interest.

2°. **Theorem 1.** The identity holds

$$\begin{aligned} \sum_{k=0}^n \omega'_k(x)\omega'_k(t) &= \frac{a_n}{a_{n+1}} \left\{ \frac{\omega'_{n+1}(t)\omega'_n(x) - \omega'_n(t)\omega'_{n+1}(x)}{t-x} + \right. \\ &+ (t-x)^{-3} [(\omega'_{n+1}(t)\omega_n(x) - \omega'_n(t)\omega_{n+1}(x) + \omega'_{n+1}(x)\omega_n(t) \\ &\left. - \omega'_n(x)\omega_{n+1}(t))(t-x) + 2(\omega_{n+1}(x)\omega_n(t) - \omega_{n+1}(t)\omega_n(x))] \right\}. \quad (3) \end{aligned}$$

We shall outline the proof. Differentiate equality (1) and then multiply it by $\omega_{k-1}(t)$. In the resulting equality replace x by t and t by x , and subtract the

second equality from the first. The subsequent reasoning is the same as in the derivation of the Christoffel-Darboux formula. The only difference is that on the right-hand side of the equality there appear sums of the form $\sum_{k=0}^n \omega'_k(x)\omega_k(t)$ and $\sum_{k=0}^n \omega_k(x)\omega'_k(t)$, which are found by differentiating formula (2), respectively with respect to x and to t .

Theorem 2. The identity holds

$$\sum_{k=0}^n (\omega'_k(x))^2 = \frac{a_n}{a_{n+1}} \left[\frac{1}{2} (\omega''_{n+1}(x)\omega'_n(x) - \omega''_n(x)\omega'_{n+1}(x)) + \frac{1}{6} (\omega'''_{n+1}(x)\omega_n(x) - \omega'''_n(x)\omega_{n+1}(x)) \right].$$

Proof. This theorem is easily derived from Theorem 1. It suffices in equality (3) to put $t = x$ and then compute its right-hand side by L' Hôpital' s rule.

Corollary 1. For any orthonormal system of polynomials $\{\omega_k(x)\}_{k=0}^\infty$ the inequality holds

$$\omega'''_{n+1}(x)\omega_n(x) + 3\omega''_{n+1}(x)\omega'_n(x) \geq \omega'''_n(x)\omega_{n+1}(x) + 3\omega''_n(x)\omega'_{n+1}(x),$$

$$n = 0, 1, 2, \dots, \quad -\infty < x < \infty.$$

Theorem 3. For any orthonormal system of polynomials $\{\omega_k(x)\}_{k=0}^\infty$ with weight $g(x)$, the equality holds

$$\sum_{k=0}^n \int_{-1}^1 (\omega'_k(x))^2 g(x) dx = \frac{a_n}{2a_{n+1}} \int_{-1}^1 [\omega''_{n+1}(x)\omega'_n(x) - \omega''_n(x)\omega'_{n+1}(x)] g(x) dx.$$

Proof. Since $\omega'''_{n+1}(x)$, $\omega'''_n(x)$ are polynomials of degrees respectively $[n - 2]$ and $(n - 3)$, by orthogonality,

$$\int_{-1}^1 \omega'''_{n+1}(x)\omega_n(x)g(x) dx = \int_{-1}^1 \omega'''_n(x)\omega_{n+1}(x)g(x) dx = 0.$$

Therefore Theorem 3 follows from Theorem 2.

Corollary 2. For any orthonormal system of polynomials $\{\omega_k(x)\}_{k=0}^\infty$ with weight $g(x)$, the inequality holds

$$\int_{-1}^1 \omega''_{n+1}(x)\omega'_n(x)g(x) dx \geq \int_{-1}^1 \omega''_n(x)\omega'_{n+1}(x)g(x) dx, \quad n = 0, 1, 2, \dots$$

Theorem 4. For the orthonormal system of Legendre polynomials $\{p_k(x)\}_{k=0}^\infty$, the equality holds

$$\sum_{k=0}^n \int_{-1}^1 (p'_k(x))^2 dx = \frac{n(n+1)^2(n+2)}{4}, \quad n = 0, 1, 2, \dots$$

We indicate the proof. From the orthogonality of the Legendre polynomials and the formula of integration by parts it follows that

$$\int_{-1}^1 [p_{n+1}''(x)p_n'(x) - p_n''(x)p_{n+1}'(x)] dx = [p_n(x)p_{n+1}''(x) - p_{n+1}(x)p_n''(x)]_{-1}^1.$$

It is not difficult to calculate that the right-hand side of this equality is equal to

$$\frac{n(n+1)(n+2)}{2} \sqrt{(2n+1)(2n+3)}.$$

Therefore Theorem 4 follows from Theorem 3.

3°. With the help of Theorem 4 one can obtain an analogue of A. A. Markov's inequality for the space L_2 of all functions square-summable on the interval $[-1, 1]$ with norm

$$\|f\|_{L_2} = \left(\int_{-1}^1 f^2 dx \right)^{1/2}.$$

Theorem 5. For any polynomial R_n of degree n , the inequality holds

$$\|R_n'\| \leq \frac{n+1}{2} \sqrt{n(n+2)} \|R_n\|_{L_2}, \quad n = 0, 1, 2, \dots \quad (4)$$

Proof. Consider the operator

$$U_n(f, x) = \int_{-1}^1 f(t) \sum_{j=0}^n p_j'(x) p_j(t) dt, \quad f \in L_2.$$

It is clear that

$$U_n(R_n, x) = R_n'(x).$$

Therefore, from the Cauchy–Bunyakovsky inequality and the orthogonality of the Legendre polynomials it follows that

$$\|R_n'\|_{L_2} \leq \left(\sum_{k=0}^n \int_{-1}^1 (p_k'(x))^2 dx \right)^{1/2} \|R_n\|. \quad (5)$$

By Theorem 4 and this inequality we have (4).

Remark 1. For $n = 1$ and $R_1(x) = p_1(x)$, equality holds in (4).

Remark 2. If the norm is defined by the formula

$$\|f\| = \left(\int_{-1}^1 f^2 g dx \right)^{1/2},$$

then the preceding arguments lead to the inequality

$$\|R_n^{(k)}\|_{L_2} \leq \left(\sum_{j=0}^n \int_{-1}^1 (\omega_j^{(k)}(x))^2 g(x) dx \right)^{1/2} \|R_n\|,$$

which is a generalization of inequality (5) to the general case of orthogonal polynomials.

4°. Let $\bar{\Omega}_n^{(k)}(L_2)$ be the set of all linear operations $V(f)$ from L_2 into L_2 having the property that $V(f, x) = f^{(k)}(x)$, if f is a polynomial of degree $\leq n$, and let $\bar{\Omega}_{n,n}^{(k)}(L_2)$ be the set of all linear operations $V(f)$ from L_2 into L_2 satisfying the conditions: 1) for any $f \in L_2$, $V(f)$ is a polynomial of degree $\leq n$; 2) if f is a polynomial of degree $\leq n$, then $V(f, x) = f^{(k)}(x)$. Obviously, $\bar{\Omega}_{n,n}^{(k)}(L_2) \subset \bar{\Omega}_n^{(k)}(L_2)$. It can be shown that $\bar{\Omega}_n^{(k)}(L_2) \neq \bar{\Omega}_{n,n}^{(k)}(L_2)$. Put

$$\bar{\rho}_{n,n}^{(k)} = \rho_{n,n}^{(k)}(L_2) = \inf_{V \in \bar{\Omega}_{n,n}^{(k)}(L_2)} \|V\|, \quad \bar{\rho}_n^{(k)} = \rho_n^{(k)}(L_2) = \inf_{V \in \bar{\Omega}_n^{(k)}(L_2)} \|V\|.$$

In the space C of all continuous 2π -periodic functions with norm $\|f\| = \max_{0 \leq x < 2\pi} |f(x)|$, the equality (1)* holds

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_{n,n}^{(k)}}{\rho_n^{(k)}} : \frac{4}{\pi^2} \ln n \right) = 1.$$

In the space C of all continuous functions on the interval $[-1, 1]$, with norm $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$, $\bar{\rho}_n^{(k)} = \bar{\rho}_{n,n}^{(k)} = T_n^{(k)}(1)$, $k = 1, 2, \dots, n$, $T_n(x) = \cos n \arccos x$ (2). A. N. Kolmogorov drew my attention to the fact that in the space L_2 of all 2π -periodic square-integrable functions with norm

$$\|f\| = \left(\int_0^{2\pi} f^2 dx \right)^{1/2}$$

the equalities $\rho_{n,n}^{(k)} = \rho_n^{(k)} = n^k$, $k = 0, 1, 2, \dots$, hold. We shall now prove a theorem concerning the space L_2 .

Theorem 6. For any $0 \leq k \leq n$,

$$\|p_n^{(k)}\| \leq \bar{\rho}_n^{(k)}(L_2) \leq \bar{\rho}_{n,n}^{(k)}(L_2) \leq \|\Pi_n\|, \quad (6)$$

$$\Pi_n(x) = \left[\sum_{j=0}^n (p_j^{(k)}(x))^2 \right]^{1/2},$$

* For the periodic case the bar over ρ is omitted.

where $\{p_j(x)\}_{j=0}^{\infty}$ are the orthonormal Legendre polynomials. In particular,

$$\bar{\rho}_n^{(n)}(L_2) = \rho_{n,n}^{(n)}(L_2) = (2n-1)!!\sqrt{2n+1}. \quad (7)$$

Proof. It is obvious that the operation

$$\bar{V}(f, x) = \int_{-1}^1 f(t) \sum_{j=0}^n p_j^{(k)}(x) p_j(t) dt$$

is of the class $\bar{\Omega}_{n,n}^{(k)}(L_2)$, and $\|\bar{V}\| \leq \|\Pi_n\|$. Therefore

$$\bar{\rho}_n^{(n)}(L_2) \leq \bar{\rho}_{n,n}^{(n)}(L_2) \leq \|\Pi_n\|. \quad (8)$$

On the other hand, for any $V \in \bar{\Omega}_n^{(k)}(L_2)$, $V(p_n) = p_n^{(k)}$, and since $\|p_n\| = 1$, it follows that $\bar{\rho}_n^{(k)}(L_2) \geq \|p_n^{(k)}\|$. Hence, together with (8), (6) follows.

If $k = n$, then, by virtue of (6), we have

$$\bar{\rho}_n^{(n)}(L_2) = \bar{\rho}_{n,n}^{(n)}(L_2) = p_n^{(n)}(x)\sqrt{2},$$

and this is equivalent to the equalities (7).

Leningrad Institute of Soviet Trade
named after Fr. Engels

Received
9 XII 1965

REFERENCES

- ¹ D. L. Berman, DAN, **138**, No. 4 (1961).
- ² D. L. Berman, DAN, **140**, No. 3 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.