

# ON THE DECOMPOSITION OF MULTIDIMENSIONAL PROBABILITY LAWS

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## Abstract

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*MATHEMATICS*

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# ON THE DECOMPOSITION OF MULTIDIMENSIONAL PROBABILITY LAWS

*(Presented by Academician Yu. V. Linnik on 25 XI 1965)*

In the theory of decompositions of one-dimensional probability laws, whose present state is set forth in the monograph of Yu. V. Linnik <sup>(1)</sup>, many profound and varied results have been obtained. Multidimensional analogues, as far as we know, have been obtained <sup>(4, 7, 13)</sup> only for Cramér' s theorem on decompositions of the Gaussian law, D. A. Raikov' s theorem on decompositions of the Poisson law, and Yu. V. Linnik' s theorem on decompositions of compositions of Gaussian and Poisson laws. The present paper is devoted to multidimensional analogues of results of D. A. Raikov <sup>(2)</sup> on holomorphic characteristic functions and of results of the author <sup>(3)</sup>, which generalize D. A. Raikov' s results <sup>(2)</sup> on decompositions of infinitely divisible laws without a Gaussian component.

We shall adhere to the following notation:  $R^{(n)}$  is real and  $C^{(n)}$  is complex  $n$ -dimensional Euclidean space;  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{t} = (t_1, \dots, t_n), \dots$  are their vectors (points);  $\operatorname{Re} \mathbf{x} = (\operatorname{Re} x_1, \dots, \operatorname{Re} x_n)$ ,  $\operatorname{Im} \mathbf{x} = (\operatorname{Im} x_1, \dots, \operatorname{Im} x_n)$ ;  $(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^n x_k y_k$ ;  $|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})}$ , if  $\mathbf{x} \in R^{(n)}$ , and  $|\mathbf{x}| = \sqrt{|\operatorname{Re} \mathbf{x}|^2 + |\operatorname{Im} \mathbf{x}|^2}$ , if  $\mathbf{x} \in C^{(n)}$ .

## 1°. Holomorphic characteristic functions.

Let  $P = P(E)$  be a probability distribution law (p.d.) on the class of Borel sets in  $R^{(n)}$ , and let  $\varphi(\mathbf{t}; P)$  be its characteristic function (c.f.):

$$\varphi(\mathbf{t}; P) = \int_{R^{(n)}} e^{i(\mathbf{t}, \mathbf{x})} dP(\mathbf{x}). \quad (1)$$

Generally speaking, the integral in (1) converges only for  $\mathbf{t} \in R^{(n)}$ , and therefore the c.f.  $\varphi(\mathbf{t}; P)$  is, generally speaking, defined only for  $\mathbf{t} \in R^{(n)}$ . If the c.f.  $\varphi(\mathbf{t}; P)$  continues as a holomorphic function into some domain  $G \subset C^{(n)}$ , then we shall denote the continued function also by  $\varphi(\mathbf{t}; P)$  and say that the c.f.  $\varphi(\mathbf{t}; P)$  is holomorphic in  $G$ .

D. A. Raikov showed <sup>(2)</sup> (see also <sup>(1)</sup>, p. 58) that if the c.f.  $\varphi(t; P)$  of a one-dimensional p.d.  $P$  is holomorphic in the disk  $|t| < r$ , then it is holomorphic

in the strip  $|\operatorname{Im} t| < r$ , and in this strip the integral in (1) converges absolutely and relation (1) remains valid. To formulate a multidimensional analogue of this theorem, recall that a convex tube domain is a domain  $H \subset C^{(n)}$  of the form  $\{\mathbf{t} : \operatorname{Im} \mathbf{t} \in B\}$ , where  $B$  is a convex domain in  $R^{(n)}$ , the base of the domain  $H$ . We shall call a domain  $G \subset C^{(n)}$  a ridge domain if it satisfies the weaker condition: from  $\mathbf{t} \in G$  it follows that  $i \operatorname{Im} \mathbf{t} \in G$ .

**Theorem 1.** If the c.f.  $\varphi(\mathbf{t}; P)$  of an  $n$ -dimensional p.d.  $P$  is holomorphic in a ridge domain  $G \subset C^{(n)}$  containing the point  $0 = (0, \dots, 0)$ , then it is holomorphic in some convex tube domain  $H \supset G$ . Moreover, throughout the domain  $H$  the integral in (1) converges absolutely and relation (1) remains valid.

This theorem cannot be improved in the following sense:

*Whatever convex tube domain  $H \subset C^{(n)}$ ,  $0 \in H$ , may be, there exists an  $n$ -dimensional probability distribution  $P$  whose characteristic function is holomorphic in  $H$  and is not holomorphic in any larger domain.*

The following problem arises: to give a complete description of the domains of holomorphy of characteristic functions that are holomorphic at the point 0. It is obvious that such domains must be symmetric in the sense that together with the point  $\mathbf{t} = \operatorname{Re} \mathbf{t} + i \operatorname{Im} \mathbf{t}$  they must contain the point  $\mathbf{t}' = -\operatorname{Re} \mathbf{t} + i \operatorname{Im} \mathbf{t}$ , and holomorphically convex. Moreover, by Theorem 1, such domains, together with every ridge domain  $G \ni 0$ , must contain the smallest tube domain  $H \supset G$ . It is possible that these three conditions also give the desired description, but we have been able to prove this only in the one-dimensional case.\*

D. A. Raikov showed <sup>(2)</sup> (see also <sup>(1)</sup>, p. 69) that if the characteristic function of a one-dimensional probability distribution  $P$  is holomorphic in the strip  $|\operatorname{Im} t| < r$ , then in the same strip the characteristic function of any component of the distribution  $P$  is also holomorphic. The multidimensional analogue of this theorem is as follows.

**Theorem 2.** *If the characteristic function of an  $n$ -dimensional probability distribution  $P$  is holomorphic in a certain convex tube domain  $H \subset C^{(n)}$ ,  $0 \in H$ , then in the same domain  $H$  the characteristic function of any component of the distribution  $P$  is also holomorphic.*

For the proof of Theorems 1 and 2 we use a device due to Cramér and Wold <sup>(8)</sup>, which consists in the fact that, for the study of a multidimensional probability distribution  $P$ , one uses a family of one-dimensional distributions—the projections of  $P$ . Recall that the projection of an  $n$ -dimensional probability distribution  $P$  onto the unit vector  $\vec{\xi} \in R^{(n)}$  is the one-dimensional probability distribution  $P_{\vec{\xi}}$  defined by the equality  $P_{\vec{\xi}}(E) = P\{\mathbf{x} : (\mathbf{x}, \vec{\xi}) \in E\}$ . Theorems 1 and 2 are obtained with the help of the following lemma.

**Lemma.** *If, for  $n$  linearly independent unit vectors  $\vec{\xi}^{(1)}, \dots, \vec{\xi}^{(n)} \in R^{(n)}$ , the characteristic functions  $\varphi(t; P_{\vec{\xi}^{(k)}})$  ( $k = 1, \dots, n$ ) are holomorphic respectively for  $-r_k^{(1)} < \operatorname{Im} t < r_k^{(2)}$  ( $r_k^{(1)}, r_k^{(2)} > 0$ ,  $k = 1, \dots, n$ ), then the characteristic function  $\varphi(\mathbf{t}; P)$  is holomorphic in the convex tube domain  $H \subset C^{(n)}$  whose base is the*

smallest convex domain in  $R^{(n)}$  containing the intervals  $\{\mathbf{x} : \mathbf{x} = \theta \vec{\xi}^{(k)}, -r_k^{(1)} < \theta < r_k^{(2)}\}$ ,  $k = 1, \dots, n$ . Moreover, throughout the domain  $H$  the integral (1) converges absolutely and relation (1) remains valid.

**2°. On decompositions of infinitely divisible laws.** Denote by  $I_0$  the class of infinitely divisible (i.d.) laws having only i.d. components. A complete description of this class is unknown even in the one-dimensional case. However, in that case there are a number of very subtle either necessary or sufficient conditions for an i.d. law to belong to  $I_0$  <sup>(1,2,10)</sup>.\*\* In the multidimensional case it is known only <sup>(4,7,13)</sup> that the Gaussian and Poisson laws, as well as their convolutions, belong to  $I_0$ . We note that multidimensional theorems on membership in the class  $I_0$ , generally speaking, cannot be obtained from the corresponding one-dimensional ones by means of the Cramér–Wold device <sup>(8)</sup>, since projections of a multidimensional law of the class  $I_0$  (for example, a Poisson law) need not belong to  $I_0$ .

We shall give two theorems containing sufficient conditions for a multidimensional i.d. law to belong to  $I_0$ . In what follows we agree to regard all sets considered in  $R^{(n)}$  as Borel sets, and all measures as defined on the class of Borel sets. We shall say that a measure  $\mu$  is concentrated on a set  $A$  if from  $E \cap A = \emptyset$  it follows that  $\mu(E) = 0$ . By  $(m)A$ , where  $A \subset R^{(n)}$ ,  $m = 2, 3, \dots$ , we agree to denote the set

$$\left\{ \mathbf{x} : \mathbf{x} = \sum_{k=1}^m \mathbf{y}^{(k)}, \mathbf{y}^{(k)} \in A, k = 1, \dots, m \right\}.$$

Let  $P$  be an i.d. law—

\* In this case, the condition of holomorphic convexity, of course, drops out.

\*\* The sufficient conditions found by Yu. V. Linnik for membership in  $I_3$  have been somewhat weakened in papers <sup>(5,6)</sup>.

By P. Lévy' s formula <sup>((9), p. 220)</sup>, we have

$$\varphi(t; P) = \exp \left\{ i(\beta, t) - Q(t) + \int_{R^{(n)}} \left( e^{i(t, x)} - 1 - \frac{i(t, x)}{1 + |x|^2} \right) d\nu_P(x) \right\},$$

where  $\beta \in R^{(n)}$ ,  $Q(t)$  is a nonnegative quadratic form;  $\nu_P$  is a completely  $\sigma$ -finite measure satisfying the condition

$$\int_{R^{(n)}} |x|^2 (1 + |x|^2)^{-1} d\nu_P(x) < \infty.$$

We shall say that the law  $P$  has no Gaussian component if  $Q(t) \equiv 0$ .

**Theorem 3.** *Let  $P$  be an infinitely divisible law without a Gaussian component, for which the measure  $\nu_P$  is concentrated in some bounded open convex set  $A \subset R^{(n)}$  having the property  $A \cap (2)A = \emptyset$ . Then  $P \in I_0$ .*

**Theorem 4.** *Let  $P$  be an infinitely divisible law without a Gaussian component, for which the measure  $\nu_P$  is completely finite and concentrated in some set  $A \subset \{x : x_1 \geq 0, \dots, x_n \geq 0\}$  having the following property: every finite system of vectors from  $A$  is linearly independent over the field of rational numbers. Then  $P \in I_0$ .*

For the proof of Theorems 3 and 4 we have somewhat refined the method applied in (3).

It follows from Theorem 3 that the class of infinitely divisible laws is an “envelope” of the class  $I_0$  in the sense that every infinitely divisible law  $P$  can be represented in the form  $P = P_1 * P_2 * P_3 * \dots$ , where  $P_k \in I_0$  ( $k = 1, 2, \dots$ ).

With the aid of Theorem 3 and result (12), one can construct an example of an  $n$ -dimensional,  $n > 1$ , infinitely divisible law  $P \in I_0$ , whose projections onto all axes in  $R^{(n)}$ , with the exception of two, do not belong to  $I_0$ . Thus, under projection the “decomposability” of an infinitely divisible law may substantially deteriorate. P. Lévy established (11) that under projection a substantial improvement of “decomposability” may also occur: there exists an indecomposable law all of whose projections are infinitely divisible laws.

**Remark.** In the one-dimensional case Theorems 3 and 4 can be strengthened. We give only the theorem strengthening Theorem 3.

**Theorem 3'.** *Let  $P$  be an infinitely divisible law whose characteristic function is representable in the form*

$$\varphi(t; P) = \exp \left\{ i\beta t + \int_{R^{(1)}} (e^{itx} - 1) d\nu(x) + \sum_{m=1}^{\infty} \lambda_m (e^{i\chi_m t} - 1) \right\},$$

where  $\beta \in R^{(1)}$ ;  $\nu$  is a completely finite measure concentrated on the interval  $[b, c]$ ,  $0 < b < c \leq 2b < \infty$ ;  $\chi_1 > c$  and the numbers  $\chi_{m+1}\chi_m^{-1}$  ( $m = 1, 2, \dots$ ) are natural numbers different from one;  $\lambda_m \geq 0$  ( $m = 1, 2, \dots$ ), and there exists a constant  $k > 0$  such that

$$\lambda_m = O(\exp\{-k\chi_m^2\}), \quad m \rightarrow \infty.$$

Then  $P \in I_0$ .

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