ON THE DEPENDENCE BETWEEN STRESSES AND STRAINS IN THE NEIGHBORHOOD OF A CORNER POINT OF THE LOADING PATH

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Abstract

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ON THE DEPENDENCE BETWEEN STRESSES AND STRAINS IN THE NEIGHBORHOOD OF A CORNER POINT OF THE LOADING PATH

Within the framework of the model proposed in paper ([1]), the question of the stress-strain state in an infinitesimal neighborhood of a kink of the loading path under plane-plastic deformation is considered. It is assumed that the kink of the path occurs after monotonic loading. The results obtained, under certain additional assumptions, are generalized to the three-dimensional case.

1°. Let an element of a body be under conditions of monotonic plane-plastic ([1]) deformation; its stress state at the time $t = t^0$ is characterized by the components $\sigma_x(t^0)$, $\sigma_y(t^0)$ ($\tau_{xy}(t^0) = 0$). The components of plastic strain $e_x(t^0)$, $e_y(t^0)$ and the intensity of slips $\varphi(\theta, t^0)$ are given by the formulas

$$e_x(t^0) = -e_y(t^0) = \frac{\sigma_0}{2k}\left(1 - \frac{\alpha_0^2}{2}\right), \qquad \varphi(\theta, t^0) = \frac{2\sigma_0}{k\pi}\sqrt{\alpha_0^2 - \theta^2},$$

$$\sigma_0 = \sigma_x(t^0) - \sigma_y(t^0), \tag{1,1}$$

where k, c, τ_s are material constants, and the parameter α_0 is determined from the equation

$$\alpha_0^2\left(\frac{1}{2} + \ln\frac{2c}{\alpha_0}\right) = \frac{1}{2} - \frac{\tau_s}{\sigma_0}. \tag{1,2}$$

Let the stresses receive arbitrary small increments $\Delta\sigma_x$, $\Delta\sigma_y$, $\Delta\tau_{xy}$ over a small time interval Δt. Suppose that at some time $t \in [t^0 + 0,\ t^0 + \Delta t]$ additional slips occur (Fig. 1) along a set of directions $\theta_0 \in [-\alpha_1(t), \alpha_2(t)]$. In the indicated directions the resistance to shear S_m and the rate of its increment are respectively equal to the shear stress τ_m and to the rate of increment of the shear stress. Thus, we have

$$k\int_{-\alpha_1(t)}^{\alpha_2(t)} \varphi_t'(\theta,t)\ln\frac{c}{|\theta-\theta_0|}\,d\theta = \frac{\partial}{\partial t}\tau_m(\theta_0,t), \quad \theta_0\in[-\alpha_1(t),\alpha_2(t)]; \qquad (1,3)$$

$$S_m^0[\theta_0,t^0]+\int_{t_0}^{t} S_m'[\theta_0,\xi,\alpha_1(\xi),\alpha_2(\xi)]\,d\xi = \begin{cases}\tau_m(\theta_0,t), & \theta_0\in[-\alpha_1(t),\alpha_2(t)],\\ \tau_m(\theta_0,t)+\eta, & \theta_0\notin[-\alpha_1(t),\alpha_2(t)],\\ & (\eta>0),\end{cases} \qquad (1,4)$$

where $\varphi_t'(\theta,t)$ is the rate of change of the slip intensity; $S_m^0[\theta_0,t^0]$ is the resistance to shear at the moment preceding the kink of the loading path; $S_m'[\theta_0,\xi,\alpha_1(\xi),\alpha_2(\xi)]$ is the rate of change of the resistance to shear after the kink of the loading path:

$$S_m^0=\tau_s+k\int_{-\alpha_0}^{\alpha_0}\varphi(\theta,t^0)\ln\frac{c}{|\theta-\theta_0|}\,d\theta, \qquad S_m'=k\int_{-\alpha_1(\xi)}^{\alpha_2(\xi)}\varphi_\xi'(\theta,\xi)\ln\frac{c}{|\theta-\theta_0|}\,d\theta. \qquad (1,5)$$

Taking the direction (θ_0) of sliding as close to the direction of action of the maximum shear stress T, for $\tau_m(\theta_0,t)$ we may set, from (1),

$$\tau_m(\theta_0,t)\approx T(t)\{1-2[\theta_0-\Phi(t)]^2\}, \qquad (1,6)$$

where

$$T(t)=\frac{1}{2}\sqrt{[\sigma_x(t)-\sigma_y(t)]^2+4\tau_{xy}^2(t)}, \qquad \Phi(t)=\frac{1}{2}\operatorname{arc\,tg}\frac{2\tau_{xy}(t)}{\sigma_x(t)-\sigma_y(t)}. \qquad (1,7)$$

Equation (1,3) and condition (1,4) determine the unknown functions $\varphi_t'(\theta,t)$, $\alpha_{1,2}(t)$. Depending on the magnitude of the ratio $\Delta(\sigma_x-\sigma_y)/\Delta\tau_{xy}$, their solution has a different character.

a) Let us first consider the case in which the kink in the trajectory occurs under conditions of monotonic plastic deformation, i.e., when the condition $\dot\alpha_{1,2}(t)>0$ $(t>t^0,\ \dot\alpha=d\alpha/dt)$ is satisfied. In this case, from (1,3) and (1,4), taking account of expression (1,6), for a small neighborhood of the corner point we obtain

$$\varphi_t'(\theta,t)=\frac{2n}{k\pi}\sqrt{(\alpha_1+\theta)(\alpha_2-\theta)}+\frac{n\tau_s+2\sigma_0\ln(2c/\alpha_0)\cdot\theta}{k\pi\sigma_0\ln(2c/\alpha_0)\sqrt{(\alpha_1+\theta)(\alpha_2-\theta)}}, \qquad (1,8)$$

$$\alpha_{1,2}(t) = \alpha_0 + \left[\frac{n\tau_s}{2\sigma_0\alpha_0\ln(2c/\alpha_0)} \mp 1\right]\frac{\Delta t}{\sigma_0}, \qquad (1,9)$$

where $n = d(\sigma_x - \sigma_y)/d\tau_{xy}$, and, since the time scale is assumed to be inessential, it is set that $d\tau_{xy}/dt = 1$.

Fig. 1

The increment of the components of plastic strain is given by the formulas

$$\Delta e_x = -\Delta e_y = K_{12}\Delta(\sigma_x - \sigma_y), \qquad \Delta e_{xy} = K_{21}\Delta\tau_{xy}; \qquad (1,10)$$

where

$$K_{12} = \frac{1}{2k}\left[\alpha_0^2\left(1 - \frac{\alpha_0^2}{2}\right) + \frac{\tau_s(1-\alpha_0^2)}{\sigma_0\ln(2c/\alpha_0)}\right], \qquad K_{21} = \frac{2\alpha_0^2}{k}\left(1 - \frac{\alpha_0^2}{2}\right). \quad (1,11)$$

On the basis of the condition $\dot\alpha_{1,2}(t) > 0$, taking account of (1,9), we obtain a restriction on the magnitude of the ratio $|d\tau_{xy}|/d(\sigma_x - \sigma_y)$ at the instant $t = t^0 + 0$ of the kink of the loading trajectory, for which formulas (1,10) are valid:

$$|d\tau_{xy}|/d(\sigma_x - \sigma_y) \leq \tau_s/2\sigma_0\ln(2c/\alpha_0). \qquad (1,12)$$

b) Violation of condition (1,12) means that one of the functions $\alpha_{1,2}(t)$ in the process of additional loading $(\Delta\sigma_x, \Delta\sigma_y, \Delta\tau_{xy})$ receives a negative increment (nonmonotonic plastic deformation). Without restricting generality, for the time being we shall assume that in the process of additional loading $\Phi(t) > 0$; then the indicated function (Fig. 1) will be $\alpha_1(t)$, so that $\alpha_1(t) < 0$ $(t > t^0)$.

In this case from (1,3) and (1,4) we obtain:

$$\varphi_t'(\theta, t) = \frac{2n}{k\pi}\sqrt{(\alpha_1+\theta)(\alpha_2-\theta)} + \frac{2 - n(\alpha_2-\alpha_1)}{k\pi}\sqrt{\frac{\alpha_1+\theta}{\alpha_2-\theta}}; \qquad (1,13)$$

$$\ln\frac{4c}{\alpha_2+\alpha_1} = \frac{4n + 8(\alpha_2-\alpha_1) - 2n(\alpha_2-\alpha_1)^2 - n(\alpha_2-\alpha_1)^2}{2(\alpha_2+\alpha_1)[4 - n(\alpha_2 - 3\alpha_1)]}, \qquad (1,14)$$

$$\alpha_2(t) = \alpha_0 + [2 - n(\alpha_0+\alpha_*)]\sqrt{\frac{\alpha_0+\alpha_*}{2\alpha_0}}\frac{\Delta t}{\sigma_0} \qquad (\alpha_* = \alpha_1(t^0+0)).$$

For the increments of the components of plastic strain we have

$$\Delta e_x = -\Delta e_y = K_{11}|\Delta\tau_{xy}| + K_{12}\Delta(\sigma_x - \sigma_y),$$
$$\Delta e_{xy} = K_{21}\Delta\tau_{xy} + K_{22}\Delta(\sigma_x - \sigma_y). \quad (1,15)$$

Here

$$K_{11} = \frac{\alpha_0 + \alpha_*}{8k}\left(4 - 5\alpha_0^2 + 2\alpha_0\alpha_* - \alpha_*^2\right),$$

$$K_{12} = \frac{\alpha_0 + \alpha_*}{64k}\left(-8\alpha_0 + 24\alpha_* + 13\alpha_0\alpha_*^2 - 27\alpha_0^2\alpha_* + 15\alpha_0^3 - 9\alpha_*^3\right), \quad (1,16)$$

$$K_{21} = \frac{\alpha_0 + \alpha_*}{2k}(3\alpha_0 - \alpha_*), \qquad K_{22} = -\frac{\alpha_0 + \alpha_*}{2k}(\alpha_0 - \alpha_*)^2,$$

where $\alpha_* = \alpha_1(t^0 + 0)$ is determined from the first equation (1,14), in which one must put $\alpha_2 = \alpha_0$.

c) In the limiting case, when $\alpha_* = -\alpha_0$, from (1,15) and (1,16) we obtain $\Delta e_x = \Delta e_y = \Delta e_{xy} = 0$.

Letting in the first equation (1,14) $\alpha_2 \to \alpha_0,\ \alpha_1 \to -\alpha_0$, we obtain the value of the ratio $|d\tau_{xy}|/d(\sigma_x - \sigma_y)$ for which the indicated limiting case occurs:

$$|d\tau_{xy}|/d(\sigma_x - \sigma_y) = -(1 - 2\alpha_0^2)/4\alpha_0. \quad (1,17)$$

When the condition

$$-(1 - 2\alpha_0^2)/4\alpha_0 \le |d\tau_{xy}|/d(\sigma_x - \sigma_y) \le 0$$

is satisfied, the increment of the components of plastic strain is equal to zero; unloading according to the elastic law takes place.

Fig. 2

2°. One of the possibilities for generalizing the obtained results to the spatial case is opened by A. A. Ilyushin' s postulate ([2]) of isotropy. In order that the plane-plastic medium considered above satisfy the indicated postulate in the space s_5, it is necessary to require fulfillment of the condition

$$\dot\sigma_z = \frac{1}{2}(\dot\sigma_x + \dot\sigma_y). \quad (2,1)$$

In this case the image of the process ([2]) lies in the plane S_1S_3 of A. A. Ilyushin' s space (s_5), and in the neighborhood of the corner point it is determined by the formulas

$$S^0 = \mathbf{i}_1 S_1^0, \qquad dS = \mathbf{i}_1 dS_1 + \mathbf{i}_3 dS_3,$$

$$\dot{\mathcal{E}} = \mathbf{i}_1 \dot{\mathcal{E}}_1 + \mathbf{i}_3 \dot{\mathcal{E}}_3, \tag{2,2}$$

where

$$S_1^0 = \frac{\sqrt{2}}{2}\sigma_0, \qquad dS_1 = \frac{\sqrt{2}}{2}(d\sigma_x - d\sigma_y), \qquad dS_3 = \sqrt{2}\, d\tau_{xy},$$

$$\dot{\mathcal{E}}_1 = \sqrt{2}\,\dot{e}_x, \qquad \dot{\mathcal{E}}_3 = \frac{\sqrt{2}}{2}\dot{e}_{xy} \quad \left(\dot{e} = de/dt, \quad S_3^0 = 0, \quad S_k^0 = dS_k = 0, \quad k = 2, 4, 5\right).$$

The magnitude and direction of the plastic-strain-rate vector $\dot{\mathcal{E}}$ (Fig. 2) are determined by the formulas

$$\dot{\mathcal{E}} = \sqrt{2(K_{11} + 2\operatorname{ctg}\beta K_{12})^2 + \frac{1}{2}(K_{21} + 2\operatorname{ctg}\beta K_{22})^2},$$

$$\operatorname{tg}\Omega = (K_{21} + 2\operatorname{ctg}\beta K_{22})/2(K_{11} + 2\operatorname{ctg}\beta K_{12}), \tag{2,3}$$

where β is the angle between the vectors S^0 and dS; $K_{ij} = K_{ij}(\alpha_0, \alpha_*)$ for the three loading regions considered above are defined above.

Consider the general case of the location of a two-link loading trajectory in the space s_5 (in Fig. 2, the broken line OA_1B_1). If the vectors S^0 and dS (Fig. 2) are specified so that the equalities

$$S = S^0, \qquad dS' = dS, \qquad \beta' = \beta, \tag{2,4}$$

hold, then the trajectory OA_1B_1, up to rotation and reflection, will coincide with the trajectory OAB. In this case, from A. A. Il' yushin' s postulate it follows that:
a) the vector $\dot{\mathcal{E}}'$ of the plastic-strain rate and the vectors S and dS lie in one plane;
b) the equalities

$$\dot{\mathcal{E}}' = \dot{\mathcal{E}}, \qquad \Omega' = \Omega \tag{2,5}$$

hold.

The projections of the vector $\dot{\mathcal{E}}'$ onto the coordinate axes are determined by the formula

$$\dot{\mathcal{E}}'_k = \left[\frac{S_k}{S}\sin(\beta' - \Omega') + \frac{dS'_k}{dS}\sin\Omega'\right]\frac{\dot{\mathcal{E}}'}{\sin\beta'} \qquad (k = 1, 2, \dots, 5), \tag{2,6}$$

where $\dot{\mathcal{E}}'$ and Ω', with account taken of (2,5), are given by expressions (2,3).

The direct relation between stresses and strains is determined on the basis of (2,6) by the known formulas for passage from vector quantities to tensor quantities. For economy of space we restrict ourselves only to the case of a plane stress state, when the element of the body is subjected to compression along the axis Ox up to $\sigma_x = -\sigma^0$, and then infinitesimal stress increments $d\sigma_x$, $d\tau_{xy}$ are imparted to it. In this case, for the components of the plastic-strain rate immediately after the break ($t = t^0 + 0$) of the loading trajectory, on the basis of (2,6) we obtain

$$\dot{e}_x = -\frac{2}{3}\left[\sqrt{3}K_{11} - \frac{d\sigma_x}{|d\tau_{xy}|}K_{12}\right], \qquad \dot{e}_y = \dot{e}_z = -\frac{1}{2}\dot{e}_x,$$

$$\dot{e}_{xy} = K_{21}\,\mathrm{sign}\,d\tau_{xy} - \frac{d\sigma_x}{\sqrt{3}\,d\tau_{xy}}K_{22} \qquad \left(\dot{e} = \frac{de}{d\tau_{xy}}\right). \tag{2,7}$$

It has been shown that, under the assumptions made above, the formulas obtained for the corner point remain valid when, in the general case, the break is preceded by a five-dimensional curvilinear loading trajectory satisfying the condition

$$\mathrm{tg}\,\beta(t) \leq \sqrt{2}\,\tau_s/2S(t)\alpha(t)\ln(2c/\alpha(t)). \tag{2,8}$$

Here $\beta(t)$ is the angle between the vector $S(t)$ and the tangent (dS/dt) to the loading trajectory; the function $\alpha = \alpha(t)$ is determined from the equation

$$\alpha^2\left(\frac{1}{2} + \ln\frac{2c}{\alpha}\right) = \frac{1}{2} - \frac{\tau_s}{\sqrt{2}S(t)}. \tag{2,9}$$

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2. A. A. Il' yushin, *Plasticity*, Publishing House of the Academy of Sciences of the USSR, 1963.

Note: Figure translations are in progress. See original paper for figures.