WELL-POSEDNESS OF THE CAUCHY PROBLEM AND ANALYTICITY OF SOLUTIONS OF AN EVOLUTION EQUATION

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Abstract

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MATHEMATICS

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WELL-POSEDNESS OF THE CAUCHY PROBLEM AND ANALYTICITY OF SOLUTIONS OF AN EVOLUTION EQUATION

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1. In a Banach space E one considers the differential equation

$$dx/dt = Ax, \tag{1}$$

where A is a closed unbounded operator with domain of definition $D(A)$ dense in E. By a solution of equation (1) on the interval $[0, T]$ is meant a continuous function $x(t)$ on $[0, T]$, with values in $D(A)$, which has, for all $t \in [0, T]$, a derivative satisfying equation (1).

The **Cauchy problem** is the problem of finding a solution for a prescribed initial condition

$$x(0) = x_0 \in D(A). \tag{2}$$

The Cauchy problem is called **well-posed** if the solution of (1)–(2) exists for every $x_0 \in D(A)$; the solution is unique and depends continuously on the initial data in the sense that from $x_n(0) \to x(0)$ it follows that $x_n(t) \to x(t)$ for all $t \in [0, T]$. In view of the time independence of the operator A, from the well-posedness of the Cauchy problem on the interval $[0, T]$ it follows that it is well posed on any finite interval $[0, T_1]$ $(T_1 > 0)$.

An operator $U(t)$ is introduced such that, for the solution,

$$x(t) = U(t)x_0 \quad (x_0 \in D(A)). \tag{3}$$

It turns out that $U(t)$ is a strongly continuous, for $t > 0$, semigroup of bounded operators.

If $x_0 \in \overline{D(A)}$, then the function $U(t)x_0$ is called a **generalized solution** of equation (1).

It is known that $\|U(t)\|$ grows at infinity no faster than an exponential (^{1}, p. 260).

Suppose now that the operator A has at least one regular point λ_0, and at it a resolvent $R(\lambda_0)$. Then the operator A has the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ for all λ with $\operatorname{Re}\lambda > \omega$, and for $x_0 \in D(A)$

$$R(\lambda)x_0 = -\int_0^\infty e^{-\lambda t} U(t)x_0 \, dt; \tag{4}$$

if, however, x is any element of E, then one may apply formula (4) to the element $x_0 = R(\lambda_0)x$ and obtain the formula

$$R(\lambda)x = R(\lambda_0)x - (\lambda - \lambda_0)\int_0^\infty e^{-\lambda t} U(t)R(\lambda_0)x \, dt. \tag{5}$$

The function $U(t)R(\lambda_0)x$ is continuous on $[0, T]$ for any $x \in E$; therefore the operators $U(t)R(\lambda_0)$ are uniformly bounded on $[0, T]$. Taking into account the behavior of $U(t)$ at infinity, we may write that

$$\|U(t)R(\lambda_0)\| \leq M_\varepsilon e^{(\omega+\varepsilon)t} \quad (t \geq 0). \tag{6}$$

From this estimate and (5) one obtains the inequality

$$\|R(\lambda)\| \leq C(1 + |\lambda|), \tag{7}$$

valid in the half-plane $\operatorname{Re}\lambda \geq \omega_2$. This estimate shows that the requirement of well-posedness of the Cauchy problem imposes rather stringent restrictions on the resolvent of the operator A. We note that from formula (4) it also follows that, for $x_0 \in D(A)$,

$$\|R(\lambda)x_0\| \to 0, \tag{8}$$

when $\operatorname{Re}\lambda \to \infty$.

2. If for equation (1) the Cauchy problem is well posed, then it will also be well posed for the equation $x' = \mu Ax$ for any $\mu > 0$. If the corresponding semigroup is denoted by $U_\mu(t)$, then obviously

$$U_\mu(t) = U(\mu t). \tag{9}$$

We now consider the equation

$$dx/dt = \xi Ax, \tag{10}$$

where ξ is a complex number.

Definition 1. The totality of all ξ for which the Cauchy problem for equation (10) is well posed will be called the **well-posedness set K_A of the operator A.**

From what was said above it is clear that this set consists of a totality of rays issuing from the point $\xi = 0$. Denote by $U_\xi(t)$ the semigroup generated by problem (10)–(2) for $\xi \in K_A$, and put $U(\xi) = U_\xi(1)$. By virtue of (9),

$$U(\mu\xi) = U_{\mu\xi}(1) = U_\xi(\mu) \qquad (\mu > 0). \tag{11}$$

Lemma 1. *If the operator A has at least one regular point, then the operators $U(\xi)$ form a semigroup in the sense that, for ξ_1, ξ_2 and $\xi_1 + \xi_2$ belonging to K_A,*

$$U(\xi_1 + \xi_2) = U(\xi_1)U(\xi_2). \tag{12}$$

The proof consists in showing, using the well-posedness of the Cauchy problem, that for $x_0 \in D(A^3)$ the functions $U((\xi_1 + \xi_2)t)x_0$ and $U(\xi_1 t)U(\xi_2 t)x_0$ give a solution of one and the same Cauchy problem for the equation $x' = (\xi_1 + \xi_2)Ax$. By uniqueness of the solution they coincide, whence (12) follows.

Theorem 1. *Under the conditions of Lemma 1, the operator-valued function $U(\xi)$ is analytic at every interior point of the set K_A.*

If ξ_0 is an interior point of K_A, then it belongs to an open sector from K_A. Using this and the semigroup property (12), one can show that, for $x_0 \in D(A)$, the function $U(\xi)x_0$ has at the point ξ_0 one and the same derivative in two non-collinear directions, and from this derive its analyticity. The analyticity of the function $U(\xi)x$ for arbitrary $x \in E$ is proved by means of its uniform approximation by analytic functions $U(\xi)x_n$, with $x_n \in D(A)$.

From the proof of Theorem 1 and known considerations it follows that

Theorem 2. *If the operator A has at least one regular point, then for every $x_0 \in D(A)$ there exists a solution $x(\xi)$ of the equation*

$$dx/d\xi = Ax(\xi) \qquad (x(0) = x_0),$$

defined and analytic in the open kernel of the well-posedness set K_A of the operator A. The exponential types of all solutions along rays belonging to a closed sector of the open kernel of the set K_A are uniformly bounded.

It is not difficult to give examples of operators A for which the well-posedness set consists only of zero. Such, for example, is an operator whose spectrum is located on the real and imaginary axes and goes to infinity in the directions $\pm\infty$ and $\pm i\infty$. If the spectrum of the operator A

lies on the imaginary axis and goes to infinity at both of its ends, then the set of correctness can consist only of points of the real axis. For a bounded operator the set of correctness coincides with the whole plane.

Theorem 3. *If the operator A is unbounded and has at least one regular point, then its set of correctness lies in some closed half-plane.*

If the spectrum of the operator A has a sequence of points λ_n tending to infinity, then one may assume that $\arg \lambda_n \to \alpha$. Then the set of correctness lies in the half-plane

$$\pi/2 + \alpha \leq \arg \zeta \leq 3\pi/2 + \alpha.$$

If, on the other hand, the spectrum of the operator A is bounded, then the space can be decomposed into a direct sum $E = E_1 + E_2$ of invariant subspaces in such a way that in E_1 the operator A is bounded, while in E_2 its spectrum is empty. If $x_0 \in E_2$ and $f \in E^*$, then the function $f(R(\lambda)x_0)$ is analytic in the entire plane. If we assume that the set of correctness contains three rays not lying in one half-plane, then for the resolvent $R(\lambda)$ estimate (7) will be valid outside some triangle. Consequently, $f(R(\lambda)x_0)$ is linear. From (8) it follows that it is equal to zero, i.e. $x_0 = 0$. Thus E_2 is empty and the operator is bounded, which contradicts the hypothesis. The theorem is proved.

3. The Cauchy problem for equation (1) is called **uniformly correct** on $[0, T]$ if it is correct and, in addition, from $x_n(0) \to x(0)$ it follows that $x_n(t) \to x(t)$ uniformly on $[0, T]$.

In order that the Cauchy problem be uniformly correct, it is necessary and sufficient that the operator A be the infinitesimal generator of a strongly continuous semigroup satisfying the C_0-condition (see (2-4)). For this, in turn, it is necessary and sufficient that the following inequalities hold for the resolvent of the operator (see (1), p. 373):

$$\|R^n(\lambda)\| \leq M/(\operatorname{Re}\lambda - \omega)^n \quad (\operatorname{Re}\lambda > \omega,\ n = 1, 2, \ldots). \tag{13}$$

Definition 2. The totality of all ζ for which the Cauchy problem for equation (10) is uniformly correct will be called the **set K_A^c of uniform correctness of the operator A.**

From conditions (13) it follows immediately that, in order that the ray $\arg \zeta = \varphi$ belong to the set of uniform correctness of the operator A, it is necessary that the inequalities

$$\|R^n(z + \eta)\| \leq M/|z - z_0|^n \quad (n = 1, 2, \ldots), \tag{14}$$

hold, where z_0 is a point on the ray $\arg z = -\varphi$, z is any point on this ray with $|z| > |z_0|$, and η is an arbitrary point on the line perpendicular to this ray. It is sufficient that conditions (14) hold for $\eta = 0$.

Lemma 2. *If two rays $\arg \zeta = \varphi_1$ and $\arg \zeta = \varphi_2$ $(0 < \varphi_2 - \varphi_1 < \pi)$ belong to the set of uniform correctness of the operator A, then every ray $\arg \zeta = \varphi$ for which $\varphi_1 \leq \varphi \leq \varphi_2$ also belongs to this set.*

From the conditions of the lemma there follows the validity of the estimates

$$\|R^n(z+\eta_1)\| \le M_1/|z-z_1|^n, \qquad \|R^n(z+\eta_2)\| \le M_2/|z-z_2|^n,$$

when z, η_1 and η_2 are situated in the manner described above. Draw at the points z_1 and z_2 perpendiculars to the rays $\arg z = -\varphi_1$ and $\arg z = -\varphi_2$. Denote their point of intersection by ξ and introduce the operator $A_1 = A - \xi I$. A simple calculation shows that for the resolvent of the operator A_1 the inequalities

$$\|z^n R^n_{A_1}(z)\| \le M_1, \qquad \|z^n R^n_{A_1}(z)\| \le M_2,$$

are valid, where z ranges respectively over the rays $\arg z = -\varphi_1$ and $\arg z = -\varphi_2$.

By the Phragmén–Lindelöf principle, then

$$\|z^n R_{A,n}(z)\| \le M = \max(M_1, M_2)$$

also inside the angle $-\varphi_2 \le \arg z \le -\varphi_1$. Hence the assertion of the lemma follows easily.

From Lemma 2 and the preceding one there follows

Theorem 4. In order that the semigroup $U(\zeta)$ corresponding to the Cauchy problem be analytic in the sector $\varphi_1 < \arg\zeta < \varphi_2$ $(0 < \varphi_2 - \varphi_1 - \pi)$ and satisfy in it the condition

$$\|U(\zeta)\| \le Me^{\omega|\zeta|}$$

it is necessary and sufficient that the rays $\arg\zeta = \varphi_1$ and $\arg\zeta = \varphi_2$ belong to the set of uniform well-posedness.

Theorem 5. The set of uniform well-posedness of the operator A can be only one of the following: 1) the point $\zeta = 0$; 2) a ray; 3) a straight line; 4) an open, half-open, or closed sector with opening angle ψ: $0 < \psi \le \pi$.

4. In conclusion we give two theorems concerning equation (1) in a Hilbert space H.

Theorem 6. In order that, for the uniformly well-posed Cauchy problem (1)–(2) in a Hilbert space H, all generalized solutions admit analytic continuation into the sector $-\varphi_0 < \arg\zeta < \varphi_0$ with the estimate

$$\|x(\zeta)\| \le Me^{\omega|\zeta|}\|x_0\| \qquad (|\arg\zeta| < \varphi_0), \tag{15}$$

where M and ω do not depend on the solution, it is necessary that the operator A admit the representation

$$A = \omega I + QB, \tag{16}$$

where Q is a positive definite self-adjoint operator, and B is a dissipative operator (see (^{4}), p. 121), for which

$$\operatorname{tg}\varphi_0 |\operatorname{Im}(Bx, x)| \leq |\operatorname{Re}(Bx, x)|.$$

Moreover, the operator $Q^{1/2}B$ has a bounded inverse defined on all of H.

Theorem 7. If the closed operator A has the representation (16), then the space H can be continuously and densely embedded in a Hilbert space $\widetilde{H}$, in which all generalized solutions of equation (1) are analytic in the sector $|\arg\zeta| < \varphi_0$, and the estimate (15) is valid (in the norm of the space $\widetilde{H}$).

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Note: Figure translations are in progress. See original paper for figures.

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