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Abstract

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G. A. SOKOLIK, N. P. KONOPLEVA

EQUATIONS OF WAVE FIELDS COVARIANT WITH RESPECT TO THE n-DIMENSIONAL REAL UNIMODULAR GROUP

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1. Earlier, one of us ([1]) made an attempt to combine Lorentz symmetry and isotopic symmetry within the framework of the group of unimodular transformations in n-dimensional real space, or, equivalently, of the unitary unimodular group in complex space. Recently this idea has acquired particular interest in connection with the successes of work on $SU(6)$-symmetry, in connection with which a number of concrete results ([2]) have been obtained. It is therefore of interest to determine the general form of the free Lagrangian of particles specified by such a symmetry, which would make it possible to find the general form of the interaction preserved by this group, i.e. the corresponding Yang–Mills field ([3]).

2. In ([4]) the general form was found for Lorentz-invariant equations, which were formulated as matrix equations of the second order with respect to spinor variables:

$$\mathcal{L}^{\alpha}_{\dot{\alpha}} \Omega_{\sigma}(\alpha) \Omega_{\dot{\sigma}}(\dot{\alpha}) \sigma_{\sigma\dot{\sigma}} \psi + m\psi = 0, \tag{1}$$

where α and $\dot{\alpha}$; $\sigma, \dot{\sigma}$ are undotted and dotted spinor indices. In this case the second-order equation reduces to equations of the type ([5])

$$\Gamma_i \partial\psi / \partial x^i + m\psi = 0, \tag{2}$$

where x_i transform according to the Lorentz group, and $\psi' = S\psi$, $S = \exp[\varepsilon^{ik} I_{ik}]$ (I_{ik} are the generators of the Lorentz group), since the latter can be represented as the product of two conjugate spinor representations $({}^1/{}_2 0) \times (0\,{}^1/{}_2) = ({}^1/{}_2\,{}^1/{}_2)$.

In other words, the second derivative with respect to the spinor variables $\partial^2/\partial\xi_\alpha \partial\xi_{\dot{\alpha}}$ belongs to the fundamental representation of the Lorentz group.

3. In ([1]) it was shown that:

$$\Gamma_i \partial\psi/\partial x_i + mI\psi = 0, \qquad \overline{\Gamma}_i \partial\overline{\psi}/\partial\overline{x}_i + m\overline{I\psi} = 0$$

(x and $\overline{x}$ are unitary spinors specified by the (10) and (01) representations, respectively, of $SU(n)$ (not covariant with respect to $SU(n)$).

Consider the second-order equation

$$B_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\psi + m\psi = 0 \qquad (i,\ j = 1,\ 2, \dots, n). \tag{3}$$

The problem of classifying equations (2) according to the representations of the real linear group A_{n-1}, isomorphic to $SN(n)$, reduces to solving the equations:

$$L^i_{i'} L^k_{k'} S^{-1} B_{ik} S = B_{i'k'}, \tag{4}$$

where

$$L^i_{i'} = \delta^i_{i'} + \varepsilon^{lp}(e_{pl})^i_{i'}, \qquad \overline{L}^k_{k'} = \delta^k_{k'} - \varepsilon^{lp}(e_{lp})^k_{k'}, \qquad S = 1 + \varepsilon^{ij}E_{ij}$$

(E_{ij} are representations of A_{n-1} [6]), $(e_{ij})^l_{l'} = \delta_{il}\delta_{jl'}$ are the generators of A_{n-1} itself).

Condition (4) leads to

$$[E_{ij}B_{ls}] = B_{is}\delta_{lj} - \delta_{is}B_{lj}. \tag{5}$$

From the form of the structure of $SU(n)$

$$[E_{ij}E_{ls}] = f^{mn}_{ij;\,ls}E_{mn},$$

where

$$f^{mn}_{ij;\,ls} = \delta_{im}\delta_{jl}\delta_{sn} - \delta_{ml}\delta_{is}\delta_{jn},$$

it follows that (5) are satisfied for $B_{ij} = E_{ij}$.

Consequently, in the case of $SU(n)$ the problem of finding covariant wave equations reduces to decomposing into irreducible products of an arbitrary representation with the regular one: $(m_1 \dots m_n) \times (10 \dots 01)$, whereas in the case of the n-dimensional generalization of the Lorentz group [4] the problem of classifying the equations $L_i \partial\psi/\partial x_i + m\psi = 0$ $(i = 1, 2, \dots, n)$ (i.e., the problem of combining relativistic and isotopic invariances) reduced to decomposing an arbitrary representation with the fundamental one: $(n = 2k + 2)$ $(m_{2k+1,\,1} \dots m_{2k+1,\,k+1}) \times (10 \dots 0)$.

Let us consider the general form

$$B_a \partial/\partial\omega_a \psi + m\psi = 0, \tag{6}$$

where $x_a = \partial/\partial\omega_a = \xi_a^i \partial/\partial x^i$ (Pfaffian linear form);

$$\xi_a^l \frac{\partial}{\partial x^l} \xi_b^k - \xi_b^l \frac{\partial}{\partial x^l} \xi_a^k = f_{ab}^c \xi_c^k.$$

Then the covariance condition for (6), with $\delta\omega^a = f_{b\ c}^{\ a} \omega^c \varepsilon^b$, has the form coinciding with (5):

$$R_a^b S^{-1} B_{bS} = B_a \tag{7}$$

or

$$[E_a B_b] = f_{ab}^c B_c.$$

Hence follows the particular solution $B_b = E_b$, where $[E_a E_b] = f_{ab}^c E_c$–generators of the substitution group $S = 1 + \varepsilon^a E_a$, i.e., in the case of $SU(n)$ the covariance criterion reduces to the requirement that the differential operator of the equation belong to the regular representation.

From $[E_a B_b] = f_{ab}^c B_c$ it follows that the direct product of an arbitrary representation of a locally compact (or compact) Lie group, i.e., a group whose structure constants are antisymmetric in all indices, with the regular representation contains among the irreducible representations into whose direct sum it decomposes the arbitrary representation itself.

From the formulas given in (6), which effectively determine the matrices E_{ij} of the generators of $SU(n)$, it follows that the expressions (5)–(7) are equivalent to the formula for decomposing the product of the regular representation of $SU(n)$ with an arbitrary one:

$$(m_1 \ldots m_n) \times (10 \ldots 01) = \sum_{1 \le i,j \le n} (m_1 \ldots m_{i\pm1} \ldots m_{j\mp1} \ldots m_n).$$

4. It is easy to show that in the case of $SU(n)$ the only possible covariant first-order equation has the form

$$B_i(x) \frac{\partial}{\partial x_i} \psi + m\psi, \tag{8}$$

where $B_i(x) = x_k L^a (E_a)_i^k$ (the parametric index may, of course, be replaced by the composite index (ik)).

In other words, in the case $L^a x_a \psi + m\psi = 0$, where $[x_a x_b] = f^c_{ab} x_c$, the differential operator has the form $x^{ik} = x_i \partial/\partial x^k$. Since we are seeking an expression for the dynamical part of the Lagrangian, from which

could derive conservation laws, i.e., an expression for the dynamical constants specifying the states of the system, the equation must be written in terms of constant matrices. Therefore $x_{ik} = x_i \partial/\partial x^k$ is replaced by the operator $\partial^2/\partial x^i \partial x^k$, which also transforms according to the regular representation.

5. In order to make clear the meaning of replacing the linear differential form $x_{ik} = x_i \partial/\partial x^k$ by $\partial^2/\partial x^i \partial x^k$, let us note that x_{ik} can be realized on Bose operators $x_{ik} = a_i^+ a_k$ (7). Then, applying $x_{ik} = a_i^+ a_k$ to the basis $a_i|0\rangle$; $a_i^+|0\rangle = 0$, and taking into account $[a_i^+ a_k] = \delta_{ik}$, we obtain: $(e_{ik})_{mn} = \delta_{im}\delta_{kn}$ (i.e., the realization of x_{ik}).

In the case of a representation of arbitrary weight, the basis of an irreducible representation of $SU(n)$ is defined in [1] as follows:

$$\begin{bmatrix} m_0 & \dots & \dots & m_{n-1} \\ & m_0 & \dots & m_{n-2} \\ & & \dots & \\ & & m_0 m_2 & \\ & & m_1 & \end{bmatrix} = \prod_{1 \le i,k \le n} \frac{a_{ik}^{m_{k-1}-m_i}}{\sqrt{(m_{k-1}-m_k)!}} \,|0\rangle \tag{9}$$

$$\begin{bmatrix} m_{n-1} m_0 & \dots & \dots & m_0 \\ m_{n-2} m_0 & \dots & \dots & m_0 \\ \dots & \dots & \dots & \\ m_2 m_0 & & & \\ m_1 & & & \end{bmatrix} = \prod_{1 \le k \le n} \frac{a_i^{m_k - m_{k-1}}}{\sqrt{(m_k - m_{k-1})!}} \,|0\rangle, \tag{10}$$

where $m_0 \ge m_k \ge m_{k+1}$; $m_{k+1} \ge m_k \ge m_0$; the weights $(m_0 \dots m_0 m_n)$ and $(m_n m_0 \dots \dots m_0)$ [1] determine the bases of the representations (9) and (10), respectively. As shown in [1], (9), (10) may be obtained from the forming polynomials

$$(u_1 a_1 + \dots + u_n a_n)^{m_0 - m_n} \times \left(\sum_{ijk=\text{cycl}} v_i a_{jk} \right)^{m_n - m_0},$$

where $a_{ik} = a_{[i} a_{k]}$, and moreover

$$\sum_{i=1}^{n} u_i v_i = 0.$$

The passage to second-order operators corresponds in this case to passing from $a_i^+ a_k$ to $e_{ik} = a_i^+ b_k^+$, where the product of annihilation operators admits the same realization as $e_{ik} = a_i^+ a_k$, if the basis of the representations has the form

$$A_{mn} = (-1)^m \frac{a^n b^{-m}}{\sqrt{n!}} \sqrt{(m-1)!}\, |00\rangle_{ab},$$

where to each class of annihilation operators there corresponds its own vacuum "ket" :

$$a_a^+ |0\rangle = 0; \qquad b_b^+ |0\rangle = 0,$$

and moreover

$$[a^+ a^n] = n_i a^{n-1}, \qquad [b^+ b^{-m}] = -m b^{-m-1}.$$

6. The introduction of a second-order operator admits an interesting geometrical analogy with Cartan' s method of exterior forms. Indeed, introducing the exterior differential operations [8]: d and δ, respectively raising and lowering the order of the forms

$$\omega = x_k E_i^k \, dx^i; \qquad d^2\omega_0 = 0; \qquad \delta\omega_0 = 0$$

(ω_0 in the present case evidently plays the role of a vacuum), we arrive at $\delta d\omega_0 = \omega_0$, and, consequently, at a realization of $SU(2)$ on the basis

$$Z_m^j = \frac{\omega_0^{j+m} (d\omega_0)^{j-m}}{\sqrt{(j+m)!(j-m)!}}.$$

Then the differential operations d and δ play the role of root vectors:

$$\binom{d}{\delta} Z_m^j = \sqrt{(j \pm m)(j \mp m + 1)}\, Z_{m\mp 1}^j, \qquad H_3 = \frac{1}{2}(d\delta - \delta d), \quad H_- = d, \quad H_+ = \delta.$$

In this case the topological Laplacian (8) $\Delta = d\delta + \delta d$ is related to the invariant, i.e. to the generalized Laplacian of $SU(2)$,

$$H^2 = H_3^2 + \frac{1}{2}(H_+ H_- + H_- H_+).$$

In the case of spinors H^2 and Δ coincide.

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Research Institute of Introscopy

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CITED LITERATURE

1. T. A. Sokolik, *Group Methods in the Theory of Elementary Particles*, 1965.

2. H. Bacry, J. Nuyts, Phys. Lett., **12**, No. 2, 156 (1964).

3. R. Utiyama, Phys. Rev., **101**, 1597 (1956).

4. D. Ivanenko, G. A. Sokolik, Nuovo Cim., No. 6, 226 (1957).

5. G. A. Sokolik, DAN, **154**, No. 3 (1964).

6. I. M. Gelfand, M. L. Tsetlin, DAN, **71**, 825 (1950).

7. M. Moshinski, J. Math. Phys., **4**, No. 9 (1963).

8. J. de Rham, *Differentiable Manifolds*, IL, 1956.

Note: Figure translations are in progress. See original paper for figures.