A PARTIALLY ORDERED SYSTEM OF ATTRACTING SETS

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Abstract

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Consider a continuous single-valued mapping T of an interval R of the real line into itself. By the attracting set Ω_x we mean the set of ω-limit points of the sequence $\{T^j x\}_{j=0}^{\infty}$, $x \in R$.

Let A be the totality of attracting sets Ω_x, as x ranges over the whole space R. The system A is naturally partially ordered: if $\Omega', \Omega'' \in A$, $\Omega' \subset \Omega''$, then Ω' precedes Ω'' in A.

In the present note the properties of the partially ordered system A are formulated.

1°. Every maximal chain of the system A has a minimal and a maximal element.

A set Ω is a minimal element if and only if, for every $x \in \Omega$, $\Omega_x = \Omega$. A minimal element may be either a finite set of points forming a cycle of the mapping T, or a nowhere dense perfect set.

A maximal element that is not minimal will be called: 1) a maximal element of the first kind if it contains no cycles, and 2) a maximal element of the second kind if it contains at least one cycle.

Every maximal element of the second kind is a perfect set; every maximal element of the first kind contains a perfect part and also a countable set of points.

A maximal element Ω of the second kind has the following important property ([1]): for any point x for which $\Omega_x \subseteq \Omega$, there exists a number j_x: $T^{j_x} x \in \Omega$, and consequently $T^j x \in \Omega$ for $j \geq j_x$.

2°. There are at most countably many maximal elements of the first and second kind.

However, there may exist a continuum of maximal elements that are also minimal.

3°. Any two maximal chains containing a common element that is not minimal have one and the same maximal element.

4°. If the mapping T has only cycles of order 2^i, $i = 0, 1, 2, ...$, then any chain consists of no more than two elements; any two maximal chains either coincide or have no common elements; every maximal element is either minimal or a maximal element of the first kind.

5°. If the mapping T has a cycle of order $\neq 2^i$, $i = 0, 1, 2, ...$, then there exists at least one maximal element of the second kind.

6°. A maximal element of the second kind has no immediately preceding element in any maximal chain.

7°. The set of minimal elements preceding a maximal element of the second kind has the cardinality of the continuum; moreover, the subset of minimal elements distinct from cycles also has the cardinality of the continuum.

8°. The set of elements immediately following each element that precedes a maximal element of the second kind has the cardinality of the continuum.

9°. For any element Ω for which the set of points x such that $\Omega_x = \Omega$ is a set of the third class in the Baire–Vallee Poussin classification, and only for such an element, there exists a chain containing Ω in which there is no element immediately following Ω.

If the mapping T has a cycle of order $\neq 2^i$, $i = 0, 1, 2, ...$, then such an element exists (1) and, consequently, the system A does not possess the descending chain condition.

The proof of assertions 1°–9° is essentially based on the results of (1). Let us also note a number of facts (10°–30°) following from (1), used for the proof of 1°–9° and of independent interest.

10°. If $\Omega', \Omega'' \in A$, the set $\Omega' \cap \Omega''$ is nonempty and at least one point of the set $\Omega' \cap \Omega''$ is limiting on the left (on the right) for the sets Ω' and Ω'', then $\Omega' \cup \Omega'' \in A$.

20°. If $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$, $\Omega_n \in A$, then the closure (in R) of the set $\bigcup_n \Omega_n$ is an element of the system A.

30°. If $\Omega' \subset \Omega$, $\Omega', \Omega \in A$ and Ω is a maximal element of the second kind, then for any point $x \in \Omega$, $x \in \Omega'$, for which $\Omega_x \subseteq \Omega'$ and Ω_x is not a cycle, there exists a set $\Omega'' \in A :\ \Omega'' \supset \Omega'$, $\Omega'' \ni x$, and moreover for any point

$$x' \in \Omega'' \setminus \Omega' \setminus \{T^j x\}_{j=0}^{\infty}$$

there is a number $m :\ T^m x' = x$, and for every m there exists only one such point x'.

In the case when T is a smooth mapping (at least continuously differentiable on R), for any element Ω not containing cycles there exists an element $\Omega' \supset \Omega$. Consequently, in this case every maximal element is either a maximal element of the second kind (a perfect set) or a minimal element (a finite set).

In (2) an example is constructed of a continuous, but nondifferentiable, mapping having a maximal element of the first kind.

It is interesting to determine to what extent the partially ordered system A characterizes the mapping. Let, for example, Ω and Ω' be maximal elements of the mappings T and T'. Further, let A/Ω, A'/Ω' be partially ordered systems consisting of attracting sets contained respectively in Ω and Ω'. Will the mappings T and T', considered respectively on the sets Ω and Ω' (i.e. $T' = STS^{-1}$, where S is a homeomorphism $\Omega \to \Omega'$), be isomorphic if the partially ordered systems A/Ω and A'/Ω' are isomorphic? Obviously, one must additionally assume that the least order of the cycles contained both in Ω and in Ω' is one and the same (if there are cycles in Ω and Ω'). It is possible that a similar result holds if one considers a partially ordered system including not all attracting sets, but only some part of the attracting sets, chosen in an appropriate way.

It is also of interest to find a complete set of properties which an arbitrary partially ordered set must satisfy in order to be isomorphic to the partially ordered system of attracting sets of some continuous mapping.

Let us note that on every maximal element Ω of the second kind the mapping T is determined by specifying, on the set $\Omega \subset R$, the system of closed sets that are attracting sets contained in Ω. Moreover, if the set Ω contains a cycle of order k (i.e. a set consisting of k points), then there exist no more than $(2k)!$ mappings, isomorphic to one another on Ω and having on Ω the given system of attracting sets.

This result is a consequence of 3^0 and of the assertions:

a) if the set Ω contains a cycle of order k, then for any point $x \in \Omega$ (except, perhaps, points of cycles of order $\leq 2k$) there exist at least two distinct points x', $x'' \in \Omega$ such that $T^{2k}x' = T^{2k}x'' = x$;

b) for any set $\Omega' \subset \Omega$, the points $x \in \Omega$ for which $\Omega_x = \Omega'$ are everywhere dense in Ω.

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Note: Figure translations are in progress. See original paper for figures.