Soviet-era science, translated into English

ON A CRITERION FOR THE CONVERGENCE OF THE GRADIENTS OF A SEQUENCE OF HARMONIC FUNCTIONS

MATHEMATICS

1966

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196601.20405

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 517.53 : 517.947.42

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. V. BITSADZE

ON A CRITERION FOR THE CONVERGENCE OF THE GRADIENTS OF A SEQUENCE OF HARMONIC FUNCTIONS

In the classical theory of functions, an important role is played by the following theorem on the convergence of a sequence $\{f_n(z)\}$, $n = 1, 2, ...$, of functions holomorphic in a domain D of the plane of the complex variable $z = x + iy$: if the sequence $\{u_n(x, y) = \operatorname{Re} f_n(z)\}$ converges to zero uniformly in the domain D, and the sequence $\{v_n(x, y) = \operatorname{Im} f_n(z)\}$ converges to zero at a fixed point $z_0 \in D$, then $\{f_n(z)\}$ converges to zero uniformly in every bounded closed domain D^* belonging to the domain D.

The present note is devoted to establishing a multidimensional analogue of this theorem.

For simplicity of notation, below we shall restrict ourselves to consideration of the three-dimensional case.

If a sequence $\{u_n(x, y, z)\}$, $n = 1, 2, ...$, of harmonic functions regular in a domain D of the space of the variables x, y, z has the properties: a) the sequence $\{\partial u_n/\partial x\}$ converges to zero in the domain D uniformly with respect to the variables x, y, z; b) the sequence $\{\partial u_n/\partial y\}$ converges to zero in the domain D uniformly with respect to the variables y, z, and c) the sequence $\{\partial u_n/\partial z\}$ converges to zero at a fixed point of the domain D, for example, at the point $(0, 0, 0)$, then the sequence $\{\operatorname{grad} u_n(x, y, z)\}$ converges to zero uniformly with respect to x, y, z in every bounded closed domain D^* lying in the domain D.

For the purpose of proving our assertion, let us note that it is always possible to indicate a positive number r such that the closed ball $C(r; x_0, y_0, z_0)$ of radius r with center at any point $(x_0, y_0, z_0) \in D^*$ will lie in the domain D.

Inside the ball $C(r; x_0, y_0, z_0)$, for each harmonic function of the sequence $\{u_n(x, y, z)\}$ the integral representation (1,2) holds

$$u_n(x,y,z) = \frac{1}{4\pi r^2} \iint_S \left[\frac{(r^2 - \xi^2 - \eta^2 - \zeta^2)(x - \xi)}{\Delta R^{1/2}} + \frac{x + \xi}{R^{1/2}} - \operatorname{Ar\,sh} \frac{x - \xi}{\Delta^{1/2}} \right] \times$$

$$\times \frac{\partial u_n(\xi,\eta,\zeta)}{\partial \xi}\, dS + \gamma_n(y,z;x_0,y_0,z_0), \tag{1}$$

where S is the sphere

$$(\xi - x_0)^2 + (\eta - y_0)^2 + (\zeta - z_0)^2 = r^2,$$

$$\Delta = (y-\eta)^2 + (z-\zeta)^2, \qquad R = (x-\xi)^2 + \Delta,$$

and $\gamma_n(y,z;x_0,y_0,z_0)$ is a completely determined regular harmonic function of the variables y, z in the cylinder

$$(y-y_0)^2 + (z-z_0)^2 < r^2.$$

Denote by δ and δ_1 positive numbers satisfying the conditions $\delta < \delta_1 < r$.

Computing the partial derivatives $\partial u_n/\partial y$ and $\partial u_n/\partial z$ from formula (1) at the points $(0,y,z)$ and $(0,0,0)$, respectively, by virtue of conditions a), b), and c), we conclude that the sequence $\{\partial\gamma_n(y,z;0,0,0)/\partial y\}$ converges to zero uniformly for $y^2+z^2 \leq \delta_1^2$, and the sequence $\{\partial\gamma_n(y,z;0,0,0)/\partial z\}$ converges to zero for $y=z=0$. Hence, by virtue of the theorem formulated at the beginning of the present note, there follows uniform convergence to zero

of the sequence $\{\operatorname{grad}\gamma_n(y,z;0,0,0)\}$ for $y^2+z^2 \leq \delta^2$. Taking this circumstance into account, on the basis of conditions a), b), and c), again from formula (1) we conclude that the sequence $\{\operatorname{grad} u_n(x,y,z)\}$ converges uniformly to zero in the ball $C(\delta;0,0,0)$.

Moving the center of the sphere $C(r;0,0,0)$ to the point (x_0,y_0,z_0) along a continuous path L lying in the domain D^*, and taking into account that condition c) may be regarded as fulfilled at the point (x_0,y_0,z_0), by repeating the argument just given we become convinced of the uniform convergence to zero of the sequence $\{\operatorname{grad} u_n(x,y,z)\}$ in the sphere $C(\delta;x_0,y_0,z_0)$. Hence, by virtue of the Heine-Borel lemma, the validity of our assertion follows immediately.

The assertion proved is, in an obvious way, rephrased for a sequence of vectors $\{P_n(u_n,v_n,w_n)\}$ that are regular solutions of the system $\operatorname{div} P_n = 0$, $\operatorname{rot} P_n = 0$.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
12 III 1966

REFERENCES

1. A. V. Bitsadze, DAN, **159**, No. 5 (1965).

2. A. V. Bitsadze, *Boundary Value Problems for Elliptic Equations of the Second Order*, “Nauka,” 1966.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.