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ON THE PROBLEM OF FRACTIONAL-LINEAR PROGRAMMING

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Abstract

Full Text

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CYBERNETICS
AND CONTROL THEORY

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ON THE PROBLEM OF FRACTIONAL-LINEAR PROGRAMMING

(Presented by Academician L. V. Kantorovich on 8 II 1966)

In the present communication we consider a problem of nonlinear programming with a fractional-linear function to be optimized:

$$\max\left\{cx+\frac{c'x}{c''x}\;\middle|\; f_i(x)\le 0,\ i\in H\right\}. \tag{1}$$

Here $x, c, c', c'' \in R_n$, c, c', c'' are linearly independent; $f_i(x)$ is convex, $i \in H$. As an example, problems of optimizing the funds of an enterprise are given.

Properties of the fractional-linear function

1. Any function

$$\varphi(y)=\alpha+cy+(\beta+c'y)/(\gamma+c''y).$$

can always be represented in the form

$$\varphi(x)=cx+c'x/c''x.$$

2. The function $\varphi(x)$ has a discontinuity or is ambiguously defined when $c''x=0$. Its derivative

$$\varphi'(x)=c+\left[(c''x)c'-(c'x)c''\right]/(c''x)^2\neq 0$$

for any $x \in R_n$. Hence $\varphi(x)$ has no stationary points and, in particular, no local extrema. It follows from this that on any set E, $\varphi(x)$ attains its extreme values on the boundary.

3. Let

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$$E_1 = \{x \mid cx \geq 0,\ c''x \geq 0\}, \qquad E_2 = \{x \mid cx \leq 0,\ c''x \geq 0\}.$$

On any segment $(x_1, x_2) \subset E_1(E_2)$, $\varphi(x)$ can attain only a local minimum (maximum). If, however, $(x_1, x_2) \subset E_1 \cup E_2$, then $\varphi(x)$ can attain on it both a minimum and a maximum.

Theorem 1. *If a closed convex set $E \subset E_1(E_2)$, then on E the fractional-linear function $\varphi(x)$: 1) has a unique minimum (maximum); 2) may have several local maxima (minima), and all of them are attained at corner points of the set E.*

Methods of solution

In the half-space $c''x > 0$, the theory of convex programming applies to problem (1) without substantial changes ([1]). Thus, in this case it can be solved by any of the feasible-direction methods. At the same time, the finding, generally speaking, of only a local extremum is guaranteed.

Let us consider in more detail the problem

$$\max\{cx + c'x/c''x \mid Ax = b,\ x \geq 0\}. \tag{2}$$

Optimality conditions

Let $\bar{x}$ be some basic feasible solution of problem (2), corresponding to the basis B. Denote

$$\delta_j = c_B B^{-1} a_j - c_j, \qquad j = 0, \dots, n; \qquad a_0 = b;$$

here c_B is the basic part of the vector c; a_j is the j-th column of the matrix A; c_j is the j-th component of the vector c. The quantities δ'_j and δ''_j, corresponding to c' and c'', are defined analogously.

Theorem 2. *Let the set of plans E of problem (2) belong to E_1. In order that the basic plan $\bar{x}$ be a point of maximum of $\varphi(x)$ on E, it is necessary and sufficient that the conditions*

$$d_j = \delta_0''^2 \delta_j + \delta_0'' \delta'_j - \delta'_0 \delta''_j \geqslant 0, \qquad j = 1, \dots, n, \tag{3}$$

be satisfied.

The simplex method. For $E \subset E_1$, problem (2) can be solved by the simplex method, using the optimality conditions (3). The only difference from the solution of a linear problem consists in the fact that in the present case, at each iteration, it is necessary to compute not only δ_j, $j = 0, \dots, n$, but also δ'_j, δ''_j, and also d_j. The pivot column is determined by $d_j < 0$ in the usual way. The other transformations remain unchanged.

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The simplex method with correction. If $E \subset E_2$, then the solution of problem (2) is unique, but, generally speaking, is not attained at a vertex. To find it one may use the simplex method with correction, the essence of which is as follows. We choose an initial approximation $x_0 \in E$ and solve the linear problem

$$\max\{\varphi'(x_0)x \mid Ax = b,\ x \geqslant 0\}.$$

Let x_1' be the solution of this problem. On the segment $[x_0, x_1']$ we determine the point x_1 at which $\varphi(x)$ attains its minimum, and take it as the first approximation. Replacing x_0 by x_1, we repeat the computations, and so on.

Remark. Conversely, the minimization problem is solved by the simplex method for $E \subset E_2$ and by the simplex method with correction for $E \subset E_1$.

Equivalence condition. Consider the problems

$$\max\{cx \mid f(x) \leqslant 0,\ x \geqslant 0\}; \tag{4}$$

$$\max\{cx + c'x/c''x \mid f(x) \leqslant 0,\ x \geqslant 0\}, \tag{5}$$

where $f(x) = (f_i(x))$. By the Kuhn–Tucker theorem, $\bar{x}$ is a solution of problem (4) if and only if there exist vectors $\bar{a}$ and $\bar{b}$ such that

$$c = \bar{a}f'(\bar{x}) - \bar{b}, \qquad \bar{a} \geqslant 0, \qquad \bar{b} \geqslant 0, \qquad \bar{a}f(\bar{x}) = 0, \qquad \bar{b}\bar{x} = 0.$$

Denote

$$\Delta(x) = [(c''x)c' - (c'x)c'']/(c''x)^2.$$

Theorem 3. *In order that the solution $\bar{x}$ of problem (4) be a solution (generally speaking, a local one) also of problem (5), it is necessary and sufficient that the condition*

$$\Delta(\bar{x}) \leqslant \bar{b} \tag{6}$$

be satisfied.

In this connection, the solution of problem (5) may be begun with the solution of the simpler problem (4). If $\bar{x}$ satisfies condition (6), or the simpler condition $\Delta(\bar{x}) \leqslant 0$, then problem (5) is solved. Otherwise we take $\bar{x}$ as the initial approximation and proceed directly to the solution of problem (5).

Applications. As an example, let us consider problems of optimizing the funds of an enterprise. Let $x = (x_j)$ be the production program, $c = (c_j)$ the price

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vector, $p = (p_j)$ the profit vector, $q = (q_j)$ the vector of circulating assets, $r = (r_j)$ the vector of fixed production assets, $j = 1, \dots, n$. It is known that the enterprise' s material-incentive fund is formed by deductions from profit under four headings:

1. For each percent of the planned increase in the volume of sales of products —amounts proportional to the wage fund.
2. For each percent of the planned increase in profitability—amounts proportional to the wage fund.
3. For quality improvement—parts of the profit obtained through quality improvement.
4. For mastering and expanding the production of new types of products—parts of the income from the sale of these products.

As a first approximation, the proportionality coefficients may be taken as independent of the magnitude of the change in the corresponding indicators. As a result we obtain

$$\varphi(x) = (cx/c_0x_0 - 1)\alpha + [px/(q+r)x - p_0x_0/(q_0+r_0)x_0]\beta +$$
$$+(p-p_0)x\gamma + p'x\delta.$$

Here x_0 is the program of the preceding year; c_0, p_0, q_0, and r_0 have analogous meanings; $p' = (p'_j)$, where $p'_j = p_j$ if the j-th product is new, and $p'_j = 0$ otherwise. We see that the size of the material-incentive fund is expressed by a fractional-linear function of the enterprise' s program.

The problem consists in maximizing $\varphi(x)$ subject to constraints on the volume of sales, product assortment, profit and profitability, production funds, wages, materials, and capacities. This is a fractional-linear programming problem with linear constraints. Its set of plans $E \subset E_1$, and, consequently, all the results obtained above are applicable here. Let us note in conclusion that the problems of optimizing the production-development fund and the fund for social-cultural measures and housing construction are of the same type.

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[1] G. Zoitendeik, *Methods of Feasible Directions*, IL, 1963. [2] A. P. Shvartsman, *Economics and Mathematical Methods*, 1, no. 4 (1965).

Note: Figure translations are in progress. See original paper for figures.

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